

MATH GIRLS³

GÖDEL'S
INCOMPLETENESS THEOREMS

$\forall \heartsuit [\exists \heartsuit']$

HIROSHI YUKI

TRANSLATED BY TONY GONZALEZ



BENTO
BOOKS

<http://bentobooks.com>

MATH GIRLS 3: GÖDEL'S INCOMPLETENESS THEOREMS
by Hiroshi Yuki

Originally published as *Sūgaku Gāru Gēderu No Fukanzensei Teiri*
Copyright © 2009 Hiroshi Yuki
Softbank Creative Corp., Tokyo

English translation © 2016 by Tony Gonzalez
Edited by Joseph Reeder and M.D. Hendon
Additional editing by Michael Klipper
Cover design by Kasia Bytnerowicz

All rights reserved. No portion of this book in excess of fair use considerations
may be reproduced or transmitted in any form or by any means without
written permission from the copyright holders.

Published 2016 by

Bento Books, Inc.
Austin, Texas 78732

bentobooks.com

ISBN 978-1-939326-27-0 (hardcover)
ISBN 978-1-939326-28-7 (trade paperback)
ISBN 978-1-939326-29-4 (case laminate)
Library of Congress Control Number: 2014958330

Printed in the United States of America
First edition, May 2016

Math Girls³:
Gödel's Incompleteness
Theorems

Contents

1	Liar, Liar	5
1.1	Who Tells the Truth?	5
1.1.1	Mirror, Mirror	5
1.1.2	Finding Truth Tellers	6
1.1.3	Same Answers	10
1.1.4	Answering with Silence	11
1.2	Logic Problems	12
1.2.1	Alice, Boris, and Chris	12
1.2.2	Thinking with Tables	13
1.2.3	Well-designed Problems	16
1.3	What Color is Your Hat?	17
1.3.1	What Others Don't Know	17
1.3.2	A Problem for the Problem Writer	20
1.3.3	Reflections	21
2	Playing Peano	23
2.1	Tetra	23
2.1.1	The Peano Axioms	23
2.1.2	Infinite Wishes	27
2.1.3	Peano's Axiom PA1	28
2.1.4	Peano's Axiom PA2	29
2.1.5	Getting Bigger	32
2.1.6	Peano's Axiom PA3	33

2.1.7	How Small is Small?	34
2.1.8	Peano's Axiom PA4	35
2.2	Miruka	38
2.2.1	Peano's Axiom PA5	40
2.2.2	Mathematical Induction	42
2.3	Infinite Walks	47
2.3.1	The Finite and the Infinite	47
2.3.2	Dynamic vs. Static	48
2.4	Yuri	49
2.4.1	What's Addition?	49
2.4.2	More Axioms Needed?	51
3	Galileo's Doubts	55
3.1	Sets	55
3.1.1	A Rude Awakening	55
3.1.2	Extensional Definitions	56
3.1.3	The Empty Set	58
3.1.4	Sets of Sets	59
3.1.5	Intersections	61
3.1.6	Unions	63
3.1.7	Inclusion Relations	65
3.1.8	Why are Sets Important?	67
3.2	Adding Logic	68
3.2.1	Intensional Definitions	68
3.2.2	Russell's Paradox	70
3.2.3	Set and Logic Operators	72
3.3	Infinity	74
3.3.1	Bijections and Birdcages	74
3.3.2	New Concepts	78
3.4	Descriptions and Expressions	80
3.4.1	Heading Home	80
3.4.2	At the Bookstore	81
3.5	Null Answer	83
4	Know Your Limits	85
4.1	At Home	85
4.1.1	Yuri	85

4.1.2	The Boy's Proof	86
4.1.3	Yuri's Proof	88
4.1.4	Yuri's Alternate Proof	89
4.1.5	My Explanation	90
4.2	At the Supermarket	92
4.2.1	Arriving at Your Destination	92
4.3	In the Music Room	96
4.3.1	Introducing Variables	96
4.3.2	Limits	97
4.3.3	Sound Makes the Music	100
4.3.4	Calculating Limits	102
5	Know Your Limits	111
5.1	At Home	111
5.1.1	Yuri	111
5.1.2	The Boy's Proof	112
5.1.3	Yuri's Proof	114
5.1.4	Yuri's Alternate Proof	115
5.1.5	My Explanation	116
5.2	At the Supermarket	118
5.2.1	Arriving at Your Destination	118
5.3	In the Music Room	122
5.3.1	Introducing Variables	122
5.3.2	Limits	123
5.3.3	Sound Makes the Music	126
5.3.4	Calculating Limits	128
5.4	Heading Home	136
5.4.1	Moving on	136
6	Leibniz's Dream	139
6.1	Yuri Implies Not Tetra	139
6.1.1	The Meaning of 'Implies'	139
6.1.2	Solving without Thinking	142
6.1.3	The Limits of Reason?	144
6.2	Tetra Implies Not Yuri	145
6.2.1	Test Anxiety	145
6.2.2	Classes	147

6.3	Miruka Implies Miruka	148	
6.3.1	In My Classroom	148	
6.3.2	Formal Systems	150	
6.3.3	Propositional Formulas	151	
6.3.4	The Form of 'Implies'	154	
6.3.5	Axioms	156	
6.3.6	Proof Theory	157	
6.3.7	Inference Rules	159	
6.3.8	Proofs and Theorems	161	
6.4	Not Me, or Me	163	
6.4.1	Good Intentions	163	
6.4.2	The Form of Form	164	
6.4.3	The Meaning of Meaning	166	
6.4.4	Implies 'Implies'?	167	
6.4.5	Asked Out	170	
7	Epsilon Delta	173	
7.1	Limits of Sequences	173	
7.1.1	Crash Course	173	
7.1.2	In the Lecture Hall	174	
7.1.3	Understanding	Complex	State-
		ments	177
7.1.4	Reading Absolute Values	179	
7.1.5	Reading 'Implies'	182	
7.1.6	Reading 'For All' and 'Some'	184	
7.2	The Limits of Functions	187	
7.2.1	Epsilon-delta	187	
7.2.2	The Meaning of (ϵ, δ)	190	
7.3	Finals	191	
7.3.1	Unranked	191	
7.3.2	Sound of Calm, Voice of Silence	191	
7.4	Continuity	193	
7.4.1	In the Library	193	
7.4.2	Continuous Nowhere	195	
7.4.3	Continuous at Only One Point	197	
7.4.4	Escape from an Infinite Maze	198	
7.4.5	Finding the Function	199	
7.4.6	Realizations	202	

8	Diagonalization	205
8.1	Sequences of Sequences	205
8.1.1	Countable Sets	205
8.1.2	Cantor's Diagonal Argument	209
8.1.3	A Challenge	218
8.1.4	Another Challenge	221
8.2	Systems of Systems	224
8.2.1	Completeness and Consistency	224
8.2.2	Gödel's Incompleteness Theorems	230
8.2.3	Arithmetic	231
8.2.4	Formal Systems of Formal Systems	233
8.2.5	Some Vocabulary	236
8.2.6	Numerals	237
8.2.7	Diagonalization Revisited	238
8.2.8	Setting Things Straight	241
8.3	Search for a Search	241
8.3.1	First Date	241
9	Between Two Solitudes	245
9.1	Overlapping Pairs	245
9.1.1	What Tetra Noticed	245
9.1.2	What I noticed	251
9.1.3	Getting Ready	251
9.2	At Home	252
9.2.1	My Own Math	252
9.2.2	Compressed Expressions	252
9.2.3	Defining Addition	256
9.2.4	In the Zone	258
9.3	Equivalence relations	259
9.3.1	The Graduation Ceremony	259
9.3.2	What Pairs Produce	260
9.3.3	From Natural Numbers to Integers	260
9.3.4	Graphing	261
9.3.5	Revisiting Equivalence Relations	266
9.3.6	Quotient Sets	269
9.4	At the Restaurant	272
9.4.1	A Pair of Wings	272
9.4.2	Loosening Up	275

10	Following Spirals	277
10.1	$\frac{0}{3}\pi$ Radians	277
10.1.1	An Irritated Yuri	277
10.1.2	Trigonometric Functions	278
10.1.3	$\sin 45^\circ$	281
10.1.4	$\sin 60^\circ$	285
10.1.5	The Sine Curve	288
10.2	$\frac{2}{3}\pi$ Radians	291
10.2.1	Radians	291
10.2.2	Teaching	293
10.3	$\frac{4}{3}\pi$ Radians	294
10.3.1	Classes Canceled	294
10.3.2	Remainders	295
10.3.3	The Lighthouse	297
10.3.4	On the Beach	298
10.3.5	Salving the Wound	300
11	Gödel's Incompleteness Theorems	303
11.1	At the Narabikura Library	303
11.1.1	Entrance	303
11.1.2	Chlorine	304
11.2	Hilbert's Program	306
11.2.1	Hilbert	306
11.2.2	Quizzes	308
11.3	Gödel's Incompleteness Theorems	312
11.3.1	Gödel	312
11.3.2	Discussion	313
11.3.3	Outline of the Proof	315
11.4	Spring: Formal System P	316
11.4.1	Elementary symbols	316
11.4.2	Numerals and Individuals	317
11.4.3	Elementary Formulas	318
11.4.4	Axioms	320
11.4.5	Inference Rules	323
11.5	Lunch Break	324
11.5.1	Metamathematics	324
11.5.2	Math on Math	324

11.5.3	Rude Awakening	325	
11.6	Summer: Gödel Numbers	326	
11.6.1	Gödel Numbers for Elementary Symbols	326	
11.6.2	Gödel Numbers for Sequences	328	
11.7	Fall: Primitive Recursion	330	
11.7.1	Primitive Recursive Functions	330	
11.7.2	Properties of Primitive Recursive Functions and Predicates	333	
11.7.3	The Representation Theorems	335	
11.8	Winter: The Long, Hard Journey to Provability	338	
11.8.1	Gearing Up	338	
11.8.2	Definitions from Number Theory	339	
11.8.3	Definitions for Sequences	342	
11.8.4	Variables, Symbols, and Formulas	344	
11.8.5	Axioms, Theorems, and Formal Proofs	354	
11.9	The New Spring: Undecidable Statements	359	
11.9.1	A Change of Seasons	359	
11.9.2	The Seeds: From Meaning to Formalism	360	
11.9.3	The Sprouts: Defining p	363	
11.9.4	The Branches: Defining r	364	
11.9.5	The Leaves: The Flow From A_1	365	
11.9.6	The Buds: The Flow from B_1	366	
11.9.7	An Undecidable Statement	367	
11.9.8	The Plums: Proof of $\neg \text{IsProvable}(g)$	367	
11.9.9	The Peaches: Proof of $\neg \text{IsProvable}(\text{not}(g))$	369	
11.9.10	The Cherry Blossoms: Proof that P is Incomplete	371	
11.10	What the Incompleteness Theorems Mean	373	
11.10.1	No Proof of Me Exists	373	
11.10.2	Overview of the Second Incompleteness Theorem	376	

11.10.3	The Fruits of the Incompleteness Theorems	379
11.10.4	The Limits of Mathematics?	381
11.11	Riding on Dreams	382
11.11.1	Not the End	382
11.11.2	What is Mine	383

Epilogue 385

Afterword 389

Recommended Reading 391

Index 401

Know Your Limits

“Now Cinderella, depart; but remember, if you stay one instant after midnight, your carriage will become a pumpkin, your coachman a rat, your horses mice, and your footmen lizards; while you yourself will be the little cinder-wench you were an hour ago.”

CHARLES PERRAULT
Cinderella, trans. Unknown

4.1 AT HOME

4.1.1 *Yuri*

“Arrrgh!” Yuri bellowed, storming into my room one Saturday in February. She threw her bag across the room, where it collided with my bookshelf.

“Whoa, what’s up with you?” I asked.

I’d heard Yuri enter my house and say hi to my mom, chipper as usual, so this display was more than a little unexpected.

“I let a boy get the best of me yesterday. Gah, I’m so mad. I *hate* losing.”

Yuri shook her head, whipping her ponytail from side to side.

“You got in a fight?”

"No way I'd lose a fight with that dweeb. He beat me in *math*—that's what's so infuriating."

Yuri pulled a notebook out of her bag, turned to a page, and slammed it on my desk.

"This problem," she said.

Problem 5-1

Is the following a true statement?

$$0.999 \dots = 1$$

"Ah, a classic," I said. "What was your answer?"

"Something along the lines of 'of course not, moron.'"

"Why'd you say that?"

"Because it *isn't*! I mean, look at it! It's zero-point-lotsa-nines. It's gotta be a little bit less than 1!"

"And what did he say?"

"He got all full of himself, said he could prove it was true."

4.1.2 *The Boy's Proof*

Yuri turned the page in her notebook to reveal a proof in a handwriting that was not her own.

Answer 5-1

Clearly, 1 equals 1.

$$1 = 1$$

Divide both sides of this equation by 3, writing the left side in decimal and the right side as a fraction.

$$0.333 \dots = \frac{1}{3}$$

Multiply both sides by 3.

$$3 \times 0.333 \dots = 3 \times \frac{1}{3}$$

Calculate both sides.

$$0.999 \dots = 1$$

Thus, $0.999 \dots = 1$.

“Not bad for a middle school student,” I said. “The kid has potential.”

Yuri jabbed my shoulder.

“You aren’t allowed to take his side! Is this right, by the way?”

“The rigor could be improved, but yeah, pretty much.”

“Bah!” Yuri slumped into the chair next to my desk. “To be honest, after I got home that day I came up with my own proof. But I was hoping it’d be wrong.”

“Why’s that?”

“Because I want the equals sign to mean things are absolutely, positively, right-on-the-nose the same! That’s what’s cool about math, that it can be *right*, no questions asked. I want to say that $0.999 \dots < 1$, none of this ‘pretty much equals’ garbage.”

“How about you show me your proof, and *then* we can talk about how all this is right or wrong.”

“I’ve got a better idea. How about I show you my proof, and you explain how it can’t be true?”

“How about we just follow the math, and see where it takes us?”

"Deal!"

4.1.3 Yuri's Proof

"Okay," I said. "Show me your proof."

"My *wrong* proof. You've just gotta show me how."

"Proceed."

"Okay, so I started thinking about how you can start with 0.9, then add a nine to get 0.99, then 0.999 and so on."

"Good so far."

"Like, 0.9 is pretty close to 1, but it's still 0.1 from getting there."

"You're talking about the difference between 1 and 0.9, yeah?"

I wrote an equation in the notebook.

$$1 - 0.9 = 0.1$$

"The difference. Exactly. Yeah, I should have used equations I guess. Here, gimme that pencil."

I handed the pencil to Yuri, and she spun the notebook toward herself.

"So there's 0.9, but then we can do 0.99."

$$1 - 0.99 = 0.01$$

"And so on and so on."

$$1 - 0.9 = 0.1$$

$$1 - 0.99 = 0.01$$

$$1 - 0.999 = 0.001$$

$$1 - 0.9999 = 0.0001$$

$$1 - 0.99999 = 0.00001$$

⋮

"If you repeat this *infinitely many times*, you end up with the difference with 1 being 0.000⋯."

$$1 - 0.999\dots = 0.000\dots$$

“On the right, you’ve got $0.000\dots$, which is just 0, right?”

$$1 - 0.999\dots = 0$$

“So if the difference between 1 and $0.999\dots$ is 0, then $0.999\dots$ must equal 1!”

$$0.999\dots = 1$$

“And that’s my proof,” Yuri said, putting the pencil down.

“That’s well thought out,” I said. “An excellent job for someone in middle school.”

“Thank you, but you really need to cut out that ‘for someone in middle school’ stuff. It’s annoying.”

“Sorry. But anyway there’s one thing we definitely have to clean up—the part where you talk about doing something ‘infinitely many times.’ That’s kind of mathematically sloppy.”

“I figured. Seems like no matter how far you go, that $0.000\dots$ is just a *liittle* bit bigger than 0. But that would mean that $0.999\dots$ is just a *liittle* bit smaller than 1, so I’m kinda hoping that’s the case.”

“Hmm... I don’t know.”

“At least give me a shot at showing you how that might work.”

“By all means.”

4.1.4 Yuri’s Alternate Proof

Yuri turned back to the notebook.

“Okay, so check this out,” she said. “Obviously 0.9 is smaller than 1, right?”

$$0.9 < 1$$

“But so is 0.99.”

$$0.99 < 1$$

“And so on and so on.”

$$0.9 < 1$$

$$0.99 < 1$$

$$0.999 < 1$$

$$0.9999 < 1$$

$$0.99999 < 1$$

$$\vdots$$

“So doesn't that show that 0.999 is less than 1?”

$$0.999 \dots < 1 \quad ?$$

Yuri put down her pencil and shrugged.

“Seems just as right as the first one, anyway. But one of them must be wrong.”

“An interesting dilemma,” I said. “On the one hand, it seems like we can say 0.9, 0.99, 0.999 \dots gets as close to 1 as you like. On the other, it seems you can say that it never gets there.”

“Well?” Yuri said, looking up at me with pleading eyes. “Which is it?”

4.1.5 *My Explanation*

I turned to a new page in the notebook, flattening the crease as I gathered my thoughts.

“Okay,” I said. “Let me start with something like what you just said, but written out a little bit different. We'll start with a sequence like this.”

$$\begin{aligned} a_1 &= 0.9 \\ a_2 &= 0.99 \\ a_3 &= 0.999 \\ a_4 &= 0.9999 \\ a_5 &= 0.99999 \\ a_6 &= 0.999999 \\ &\vdots \\ a_n &= 0.\underbrace{9999 \dots 9}_{n \text{ 9s}} \\ &\vdots \end{aligned}$$

Yuri nodded. “So you're naming them all as a with a subscript. Got it.”

“Also, the subscript shows how many 9's there are. So that would seem to present this dilemma.”

- (1) The more 9's there are, the closer a_n is to 1.
- (2) No matter how big n is, a_n is less than 1.

"Yes, exactly," Yuri said. "One of those has to be wrong. Right?"

"Well, no. These are both true statements."

"Huh? So $0.999\cdots$ is less than 1 after all? But I thought you said—"

"No, that's not right, either. $0.999\cdots = 1$ is a true statement, too."

"I am *so* confused," Yuri said.

She frowned as she concentrated on what we'd written. I left her alone to give her some time to ponder it all. There are few things more important to doing math than time to think. I thought about changing my slogan to 'Silence is the key to—'

A loud clatter of pans came from the kitchen. Apparently my mother was cooking again.

Yuri perked up with a grin.

"I've got it! We just have to change the definition of 'equals,' so that teeny tiny differences don't matter! Mathematicians change definitions all the time, right?"

It took me a moment to recover from the audacity of her suggestion.

"That's an amazing answer, Yuri. So amazing that I wish it were true. But we can't change what equals means. It really does mean absolutely, positively, right-on-the-nose the same."

Yuri's face fell.

"I give up, then. I don't get it."

Just then, a wail of despair came from the kitchen. Yuri and I ran downstairs to see what had happened.

"What's wrong?" I asked my mother, who was standing in front of the open refrigerator in an apron. She turned to me, pale and wide-eyed.

"We're out of eggs! I'd forgotten that I used them all last night!"

"Seriously? That's *it*?"

"But—but I was going to make omelettes!"

"Can't you make something else?"

"I've already got everything else prepared! I..." Mother's expression softened, and the tone in her voice became cloyingly sweet. "Honey, I know you're studying and all, but I don't suppose—"

"Awww, mom. It's freezing out there!"

"C'mon," Yuri said, laughing, "I'll go with you."

4.2 AT THE SUPERMARKET

4.2.1 *Arriving at Your Destination*

We rode my bicycle to the supermarket, Yuri standing on the rear hub. I was thankful for the exercise to keep warm—it was even colder than I'd feared.

After a short search we found the eggs, paid at the register, and started back. Just as we were heading for the door, Yuri snagged my arm.

"Ooh, look what they've got!" She was pointing at the in-store snack counter. "Ice cream!"

"We've got to get back, Yuri. Mom's waiting for the eggs. Besides, ice cream in this weather? Are you insane?"

"Ice cream is one of those delicious-all-year-round foods," she said, maneuvering in front of me to cut off my escape route.

"Pleeease?" she said, making sad puppy eyes.

I sighed in resignation. We bought two vanilla cones and sat at the counter to eat them.

"Happy now?" I asked.

"Perfectly," Yuri said with a huge smile.

"At least one of us is," I grumbled. "And at least you aren't raging like you were before."

"I have no idea what you're talking about. Delicate flowers do not 'rage.'"

We sat there, licking our ice cream for a while.

"So what do you want to be when you grow up?" Yuri asked.

"I dunno. How about you?"

"Hmm... A lawyer, maybe."

"Sounds like you've been watching too many crime dramas lately."

"That's—that's possible. It would still be pretty cool, though."

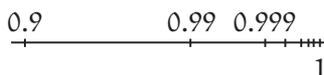
Yuri turned her cone in her hand, checking for wayward rivulets of ice cream. I pulled a mechanical pencil out of my breast pocket and looked around for something to write on. I settled on the back of a nearby store flier.

“Would it bug you if your wife made more money than you?” Yuri asked.

“Huh? Where did that come from?” I said, sketching a graph.

“Just wondering. I guess you wouldn’t care. You’re too oblivious to what people think.”

“Whatever. Here, check this out,” I said, pushing the graph toward her. “This is what we were talking about before.”



“I know,” Yuri said.

“See how this sequence 0.9, 0.99, 0.999 is getting arbitrarily close to 1? It can get as close as you want it to be, because its destination is 0.999 . . .”

“I said, *I know*. I understand that the sequence can get as close to 1 as you want it to. I get it. Done.”

“So what’s the problem?”

“That *closer* ain’t *is*. Your explanations still aren’t showing me how 0.999 . . . and 1 can be the same.”

Glowing, Yuri took a joyless lick at her ice cream.

“Okay, how about this. I’m going to ask you some questions, you answer yes or no.”

“Fine.”

“If you continue the sequence 0.9, 0.99, 0.999 on and on, will you eventually reach 1?”

“No. It doesn’t matter how many nines you add, you still aren’t there.”

I nodded. “That’s correct.”

Yuri made a scary growling noise.

“You’re *trying* to making me mad, aren’t you.”

"Hang on, hang on. I'm not done yet. One more question: as you continue the sequence 0.9, 0.99, 0.999, is there some number that you're getting closer and closer to?"

"Yes, you're getting closer to 1. We've been through this like a million times."

"Right, but let me add one more thing. A notational rule, a way we use symbols to represent an idea."

"Okay, what's the rule?"

"That when we have a sequence like 0.9, 0.99, 0.999, one that can get arbitrarily close to some number, we represent that 'some number' like this."

0.999...

Yuri froze mid-lick, her eyes wide, staring at what I'd written.

She held up a hand and said, "Hold up a minute. So this is..."

I waited while she did some internal processing. Harsh winter sunlight gleamed gold in her hair.

"Okay," she said, "I think I finally get this. Let me make sure."

"Absolutely."

"This 0.999..., it's a representation of *some number*."

"Right."

"And that *some number* that it's representing is the number that it's getting closer and closer to."

"That it gets arbitrarily close to, yes."

"Arbitrarily close to, then. But because it never gets there, that *some number* never shows up."

"That's right."

"And that *some number* that 0.999... represents is none other than 1."

"Exactly."

Yuri sighed.

"Okay, not only do I understand this now, I now understand what I didn't understand."

"Namely?"

"That 0.999... isn't a number. Not the kind I'm used to, at least. It's a representation of a number."

Yuri jotted down some notes, pausing to catch escaping streams of melting ice cream as she did.

- $0.999\dots$ represents *some number*.
- Continuing the sequence $0.9, 0.99, 0.999$, we can get arbitrarily close to that *some number*.
- But we never reach that *some number*, so it doesn't show up in the representation.
- The *some number* that $0.999\dots$ represents is equal to 1

Yuri worked her way down the length of the cone as she reviewed this. When the cone had vanished, she gave a firm nod.

"This ' $0.999\dots$ ' has got to go. It's confusing," she said.

"How so?"

"Well think about it. Say you're writing a sequence of numbers like this."

$0.9, 0.99, 0.999, \dots$

"You add those three dots at the end to mean 'goes on forever,' yeah? That made me think that when we were playing with $0.9, 0.99, 0.999$, a $0.999\dots$ would eventually show up. But it doesn't, does it? There's no $0.999\dots$ at the end of $0.9, 0.99, 0.999$ somewhere. We need a new way of writing this. Something like..."

- $0.9, 0.99, 0.999, \dots$ becomes arbitrarily close to \heartsuit .
- \heartsuit is therefore equal to 1.

"This would make things so much clearer."

I nodded. "I agree."

"But it's *your Fault. Too*," Yuri said, punctuating her words with finger jabs at my chest. "You should have told me from the beginning that $0.999\dots$ was a notational whatever, not a normal number. This problem isn't really about math, it's just about how you write things!"

"Yeah, I guess you're right. Glad to see you get it now, though."

"Well at least I didn't see this at school first. There's no way I'd have figured out what was going on. That $0.999\dots$ isn't some number that's eventually going to show up in a sequence, it's the destination where the sequence is heading, even if it never gets there.

And *that's* why $0.999\dots$ and 1 are absolutely, positively, right-on-the-nose the same."

I nodded again, smiling.

"Oh," Yuri said, "I just realized something else. These two numbers are different too, right?"

$0.999\dots$ (equal to 1)

$0.999\dots 9$ (less than 1)

"They are. The three dots at the end of a number show where the number is heading. The three dots in the middle of a number is just an abbreviation. The first one never shows up in the sequence, the second one will, eventually. They're completely different things."

"They're way confusing things, is what they are."

"I know you'll never confuse them again, though."

I looked down, and saw a white plastic bag at my feet. It took me a moment to realize what I was looking at.

"The eggs! Mom's still waiting for us!"

Answer 5-1

The following is a true statement.

$$0.999\dots = 1$$

4.3 IN THE MUSIC ROOM

4.3.1 *Introducing Variables*

"You think it was Yuri's boyfriend that gave her that problem?" Tetra whispered.

We were hanging out in the music room after classes. Seated next to Miruka at the piano was her friend Ay-Ay, president of the school piano club "Fortissimo." Ay-Ay was in eleventh grade, like Miruka and me, but in a different homeroom. She was into music like I was into math, and spent most of her free time here, practicing. The music room was normally locked up after hours, but she was so talented that the music department head had given her a key.

We watched Miruka and Ay-Ay's backs as they played, Miruka's long, straight hair in sharp juxtaposition to Ay-Ay's wavy locks. They took turns playing pieces and argued between each. Ay-Ay would insist that they differentiate between 'mechanical Bach' and 'celestial Bach,' while Miruka argued the need to extract the 'formal Bach' from the 'meta-Bach.'

I had not a single clue what they were talking about.

"No way," I whispered back to Tetra, knowing that Miruka was in no mood to have her music interrupted.

"Betcha he is," Tetra said, a sly smile on her face. "Or wants to be. He's trying to impress her. It's a sign of affection."

I brushed away her words with a gesture.

"Yuri sure is smart," Tetra continued, a hint of admiration in her voice. "I still can't shake the feeling that $0.999 \dots$ is a little bit less than 1."

Tetra pulled her notebook and a pencil out of her bag.

"You said that when you explained all that to Yuri, you used n nines in the decimal. Like this, right?"

$$a_n = 0.\underbrace{999 \dots 9}_n$$

"Sure. Using variables like that can make explanations a lot clearer. In this case two variables, I guess—the a and its subscript n . Anyway, doing that's a lot easier than saying 'the number of nines in the number $0.999 \dots 9$.'"

"I can see that. I need to get a lot more comfortable with introducing variables like this. It doesn't come natural to me at all. Seeing more letters still just makes everything look more confusing."

Tetra began writing letters in her notebook, as if practicing.

I looked back to the piano, where it was Ay-Ay's turn to play. Miruka had gotten up and was standing behind Ay-Ay with her arms crossed. She glanced back at me but immediately returned her gaze to the keys.

4.3.2 Limits

I motioned Tetra back to a further corner of the music room, where we could speak in more normal voices.

"Let's talk about limits, then," I said. I held out my hand, and Tetra passed me the notebook and pencil.

"Say you have some number a_n , and the bigger you make n , the closer a_n gets to some certain number. In a situation like that, the number you're getting close to is called a limit. You write it like this."

$$\lim_{n \rightarrow \infty} a_n$$

"Let's use A to name the number that a_n is getting closer and closer to. Then you can say that this limit equals A , which you write like this."

$$\lim_{n \rightarrow \infty} a_n = A$$

"You can also write it without using the limit notation, like this."

$$a_n \rightarrow A \quad \text{as} \quad n \rightarrow \infty$$

"A couple more words to learn. When a sequence can get arbitrarily close to some number, you say it *converges* to that number. So saying that a sequence converges is the same as saying it has a limit. Also, finding the limit of some sequence is often called *taking* the limit."

The limit of a sequence

a_n gets arbitrarily close to A as n increases

$$\iff \lim_{n \rightarrow \infty} a_n = A$$

$$\iff a_n \rightarrow A \quad \text{as} \quad n \rightarrow \infty$$

$$\iff \text{sequence } \langle a_n \rangle \text{ converges to } A$$

I watched Tetra's eyes as she read what I'd written. She pointed at a line.

"How do you read this?"

$$a_n \rightarrow A \quad \text{as} \quad n \rightarrow \infty$$

“You’d say ‘ a_n approaches A as n approaches infinity.’”

“And this one?”

$$\lim_{n \rightarrow \infty} a_n = A$$

“I’d read that, ‘the limit of a_n as n approaches infinity is A .’”

“Hmm . . . I think the first one is easier to understand, but the second one is the right one, right?”

“It isn’t more correct, but it’s what you’ll see most often. It’s more compact, if nothing else.”

Tetra reflected on this for a moment, her eyebrows drawing in.

“Since we’re talking about things getting closer, why the equals sign? Shouldn’t it be this?”

$$\lim_{n \rightarrow \infty} a_n \rightarrow A \quad ?$$

“Ah, interesting,” I said. “But no, the arrow shows change. We want to use it in the $n \rightarrow \infty$ part, to show that n is getting bigger and bigger. But if we wrote $\lim_{n \rightarrow \infty} a_n \rightarrow A$, that would mean the limit of a_n is getting closer to A .”

“Isn’t it?”

“No. Watch out for that. The limit $\lim_{n \rightarrow \infty} a_n$ is a specific number, the number that $\langle a_n \rangle$ converges to. That doesn’t change.”

$$\begin{array}{ll} \lim_{n \rightarrow \infty} a_n \rightarrow A & \text{incorrect} \\ \lim_{n \rightarrow \infty} a_n = A & \text{correct} \end{array}$$

“Oops, sorry, I’ve got it straight now.” Tetra said, blushing. She looked back at the notebook. “Oh, one more thing. Not all sequences will converge to something, right?”

“Absolutely not. What does a sequence like this do?”

$$10, 100, 1000, 10000, \dots$$

“It keeps getting bigger and bigger and bigger.” Tetra spread her arms wider and wider to illustrate.

“That’s right, and it never stops getting bigger. There’s no way it will get closer and closer to some specific number. We say a sequence like this doesn’t *converge*, it *diverges*. This particular sequence never stops getting bigger, so we can say that it diverges to positive infinity.”

“We can’t say that it *converges to* positive infinity?”

“No. Positive infinity isn’t a number, so you can’t get arbitrarily close to it. So you can never say that a sequence has infinity as a limit, or that it converges to infinity. You can just say that it diverges to positive infinity—or negative infinity, if it’s heading that way.”

“Okay, got it.”

4.3.3 *Sound Makes the Music*

I heard voices coming toward us.

“I’m telling you, the C \sharp just doesn’t work there,” Ay-Ay was saying.

“Oh, I don’t know . . .” Miruka said.

“It breaks the pattern!”

“Maybe that’s why my fingers don’t seem to want to hit it.”

The two walked up to where we had taken refuge. Ay-Ay’s face was dark.

“Taking a break?” I asked.

“What are you two talking about?” Miruka said, ignoring me.

“Limits!” Tetra said. “This is some tricky stuff.”

“Oh?” Miruka said, cocking her head.

“Well, I think I get the idea of getting arbitrarily close to something, but when you put it all in symbols I’m not so sure any more. I lose my instinct for it, somehow.”

Ay-Ay stepped forward and inserted herself into the conversation.

“Listen,” she said, “I’m no mathematician, but the way I see it, you use all those symbols in math because that’s the best way to say what needs to be said.”

Ay-Ay held out her hands and examined her palms. She turned them over and considered their backs for a time. I regarded her impressively long, strong-looking fingers—the fingers of a true pianist.

“Music is all about sounds,” she continued, still looking at her hands. “Just . . . sounds.” Her tone was uncharacteristically serious.

“Sometimes you can describe the world using words. When you can do that, great. Have at it. But there’s also a world that can only be described with sound.”

She formed a hand into a fist, thumb extended, and used that to point to her chest.

“Music is *mine*. It’s what I use to let loose feelings that would rip me to shreds if I kept them pent up. It’s all I *can* use. So I eat for music, I breathe for music, I live for music.”

Ay-Ay’s solemn air and grim expression left no room for response.

“Sometimes I meet people who say they ‘don’t get’ music,” she said. “They’re usually the kind of person who can’t ‘get’ anything they can’t put into words, because music demands understanding on its own terms. Music isn’t about words, it’s about sounds. It can only *be* sounds. If you think you can express it in words, you aren’t really listening. If you’re searching for words, you aren’t experiencing the performance, you aren’t hearing the music. You’re in a different time and space, where the music isn’t. I want to scream at people like that, ‘Stop looking for words! Open your ears!’”

Ay-Ay paused, took a deep breath, and looked at Tetra.

“If you try to study math without reading the equations, without *really* reading them, aren’t you doing the same thing?”

Tetra let out a small gasp.

“Wow, I totally see what you mean,” Tetra said. “If you don’t read equations, you aren’t seeing the world that mathematics is presenting. I guess if you try to stick to words without embracing the math, you aren’t really *doing* math, you’re just kind of watching it.”

“Music and mathematics seem like completely different things on the surface,” I said, “but in a lot of ways they’re really similar.”

Ay-Ay nodded.

“Musicians describe a world of music, so shut up and listen to the sounds. Mathematicians describe a world of mathematics, so shut up and embrace the equations.”

Tetra smiled. “So sounds are the words of music, and equations are the words of mathematics!”

“It’s always back to words with this girl,” Ay-Ay grumbled.

"Oh, not literally!" Tetra rushed to add. "I just mean . . . as a basic unit of representation."

"It's not only equations, though," I said. "It's concepts, too. Like, when we talk about limits, we say a value gets 'arbitrarily close to' some other number, not that it reaches it. I think a deep understanding of concepts like that are vital to understanding the equations that represent them."

"In any case," Ay-Ay said, "I'll stick to my music. I don't know if I'll be able to get a job related to music after I graduate, but it's something that will be part of me for the rest of my life. Absolutely."

Ay-Ay slapped her hands together, shattering the somber mood she'd created.

"Why're you guys all acting so *serious*?" she said, laughing. "Time to lighten up."

I smiled at her.

"I don't think you have anything to worry about," I said, "about working with music and all. Your playing is amazing, and your compositions are genius. I can't wait to see what you go on to do."

Ay-Ay pounded me on the back.

"You're a good kid," she said. "For a math nerd."

4.3.4 *Calculating Limits*

Ay-Ay left the room, saying she needed a break. The purely musical element of our set now gone, our talk soon turned to math.

"So did you show Tetra how to take some basic limits?" Miruka asked me.

"Like what?" I asked.

"Like this," she said, helping herself to my notebook and pencil.

Problem 5-2 (Basic limits)

$$\lim_{n \rightarrow \infty} \frac{1}{10^n}$$

"Uh, well . . . no?" Tetra said, turning panicky eyes my way.

"Give me a shot at this?" I offered.

“Be my guest,” Miruka said, smiling as she handed back the pencil and paper.

“Okay, Tetra. So what we want to find is the value of this expression.”

$$\lim_{n \rightarrow \infty} \frac{1}{10^n}$$

Tetra nodded. “So we’re looking for the limit of $\frac{1}{10^n}$ as n heads off toward infinity, right?”

“Right. In terms like you suggested before, we’re looking for the value of the ‘club’ in this expression.”

$$\frac{1}{10^n} \rightarrow \clubsuit \text{ as } n \rightarrow \infty$$

“Got it! How do we start?”

“As always, with an explicit representation of this sequence. Like I always say, examples are the key to understanding. Anyway, the sequence looks like this.”

$$\frac{1}{10^1}, \frac{1}{10^2}, \frac{1}{10^3}, \frac{1}{10^4}, \frac{1}{10^5}, \dots, \frac{1}{10^n}, \dots$$

“What we want to know,” I continued, “is if $\frac{1}{10^n}$ is heading for some specific value as n gets bigger and bigger. And if it is, we want to know what that number is.”

“Hmm . . .” Tetra said, her eyebrows knitting in concentration.

“Let me give you a hint—for now, just pay attention to the denominators.”

$$10^1, 10^2, 10^3, 10^4, 10^5, \dots, 10^n, \dots$$

“Oh,” Tetra said. “So they’re increasing like this?”

$$10, 100, 1000, 10000, 100000, \dots, 10^n, \dots$$

“That’s right. So as n gets bigger, what happens to 10^n ?”

“It gets bigger, too. Like, *way* bigger.”

“Exactly. That means we can say this.”

$$10^n \rightarrow \infty \text{ as } n \rightarrow \infty$$

I tapped what I'd written with my pencil.

"This means that as n gets bigger, the denominator of $\frac{1}{10^n}$ gets bigger, without limit. So what will happen to the fraction $\frac{1}{10^n}$ as its denominator keeps increasing?"

"The fraction . . . should get smaller and smaller, right?"

"Exactly. In fact, we can make it arbitrarily close to 0, just by making n big enough. That means we can say this."

$$\frac{1}{10^n} \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

"Putting this into the standard form for limits, we get this."

$$\lim_{n \rightarrow \infty} \frac{1}{10^n} = 0$$

"So we've shown that the limit exists, and its value is 0."

Answer 5-2 (Basic limits)

$10^n \rightarrow \infty$ as $n \rightarrow \infty$, so $\frac{1}{10^n} \rightarrow 0$. We therefore have that

$$\lim_{n \rightarrow \infty} \frac{1}{10^n} = 0.$$

"Huh . . ." Tetra said, chewing on a nail as she pondered this. "One question?" she said.

"You bet."

"This came up in how you explained the problem, right?"

$$10^n \rightarrow \infty \quad \text{as } n \rightarrow \infty$$

"It did."

"And does that mean we can write this?"

$$\lim_{n \rightarrow \infty} 10^n = \infty$$

"Yep. Something wrong with that?"

"Well, it's just that—maybe I'm not understanding this right, but doesn't this mean you're saying that the limit of 10^n here is infinity?"

“Sure.”

“But didn’t you also say that we can’t say the limit of a sequence is infinity? Just that it diverges to infinity?”

“Ah, right. Sorry, I didn’t explain that well enough. You’re right to be suspicious here, since infinity isn’t a number. But what’s going on is that we’ve expanded the definition of the = operator to mean this.”

$$\lim_{n \rightarrow \infty} 10^n = \infty \iff 10^n \rightarrow \infty \text{ as } n \rightarrow \infty$$

“Oh, okay” Tetra said. “So $\lim_{n \rightarrow \infty} 10^n = \infty$ is another way of saying that the sequence $\langle 10^n \rangle$ diverges to infinity.”

I nodded. “Yes, that’s right.”

An impatient Miruka broke her silence.

“Next problem.”

Problem 5-3 (basic limits)

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{10^k}$$

Tetra looked back and forth between this problem and the previous one.

“Um . . . is this not the same thing as what we just did?” she asked.

Miruka grimaced. “Who was it that was just going on about how not reading equations prevents you from seeing the world that mathematics is presenting?”

“Okay, let me read this one more time, carefully.” Tetra pulled the notebook closer and peered at the problem. “Oh, I get it,” she said. “The sigma makes all the difference, doesn’t it. I have no idea how to solve this, though. How do you take the limit of a sum?”

“Have fun,” Miruka said, patting me on the shoulder. She turned and headed back to the piano.

“Okay,” I said. “Here’s what we want to solve.”

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{10^k}$$

“To do that, first we need to pay attention to exactly what it is we’re taking the limit of.”

$$\sum_{k=1}^n \frac{1}{10^k}$$

“Let’s think about how we can represent this as an expression involving just n . That’ll be a lot easier than dealing with the sigma. So what’s the first thing to do to make sure you understand what’s going on here?”

“I know! Write out some examples!”

Tetra took the pencil and started writing.

$$\sum_{k=1}^1 \frac{1}{10^k} = \frac{1}{10^1} \quad (\text{for } n = 1)$$

$$\sum_{k=1}^2 \frac{1}{10^k} = \frac{1}{10^1} + \frac{1}{10^2} \quad (\text{for } n = 2)$$

$$\sum_{k=1}^3 \frac{1}{10^k} = \frac{1}{10^1} + \frac{1}{10^2} + \frac{1}{10^3} \quad (\text{for } n = 3)$$

“Very good,” I said. “Can you use this to write a general expression?”

“I think so . . . Yeah, I can!”

$$\sum_{k=1}^n \frac{1}{10^k} = \frac{1}{10^1} + \frac{1}{10^2} + \frac{1}{10^3} + \cdots + \frac{1}{10^n} \quad (\text{general expression})$$

“Good job—now we’ve established the groundwork for finding this limit. Next we want to massage this into a more useful form, one with the terms shifted. We can do that by multiplying both sides of the equation by $\frac{1}{10}$.”

$$\sum_{k=1}^n \frac{1}{10^k} = \frac{1}{10^1} + \frac{1}{10^2} + \frac{1}{10^3} + \cdots + \frac{1}{10^n} \quad \text{general expression}$$

$$\frac{1}{10} \cdot \sum_{k=1}^n \frac{1}{10^k} = \frac{1}{10} \cdot \left(\frac{1}{10^1} + \frac{1}{10^2} + \frac{1}{10^3} + \cdots + \frac{1}{10^n} \right) \quad \text{mult. both sides by } \frac{1}{10}$$

$$\frac{1}{10} \cdot \sum_{k=1}^n \frac{1}{10^k} = \frac{1}{10} \cdot \frac{1}{10^1} + \frac{1}{10} \cdot \frac{1}{10^2} + \frac{1}{10} \cdot \frac{1}{10^3} + \cdots + \frac{1}{10} \cdot \frac{1}{10^n} \quad \text{expand the right side}$$

$$\frac{1}{10} \cdot \sum_{k=1}^n \frac{1}{10^k} = \frac{1}{10^2} + \frac{1}{10^3} + \frac{1}{10^4} + \cdots + \frac{1}{10^{n+1}} \quad \text{expr. with terms shifted}$$

Tetra ran a finger down each line, confirming what I was doing in each step.

“By ‘shifted terms,’ you mean that the exponents on the 10s are each increased by 1, right?”

“That’s right,” I said. “Now we can subtract this shifted equation from the generalized equation. That will kill off all the intermediate terms.”

$$\begin{array}{r}
 \sum_{k=1}^n \frac{1}{10^k} = \frac{1}{10^1} + \frac{1}{10^2} + \frac{1}{10^3} + \cdots + \frac{1}{10^n} \quad \text{general expr.} \\
 - \quad \frac{1}{10} \cdot \sum_{k=1}^n \frac{1}{10^k} = \frac{1}{10^2} + \frac{1}{10^3} + \cdots + \frac{1}{10^n} + \frac{1}{10^{n+1}} \quad \text{shifted expr.} \\
 \hline
 \left(1 - \frac{1}{10}\right) \cdot \sum_{k=1}^n \frac{1}{10^k} = \frac{1}{10^1} - \frac{1}{10^{n+1}} \quad \text{difference}
 \end{array}$$

“Oh, neat!” Tetra said. “Everything except the first and last terms went away!”

“Let’s calculate this and see what we get.”

$$\begin{aligned}
 \left(1 - \frac{1}{10}\right) \cdot \sum_{k=1}^n \frac{1}{10^k} &= \frac{1}{10^1} - \frac{1}{10^{n+1}} && \text{previous equation} \\
 \frac{10-1}{10} \cdot \sum_{k=1}^n \frac{1}{10^k} &= \frac{1}{10^1} - \frac{1}{10^{n+1}} && \text{calculate left side} \\
 \frac{9}{10} \cdot \sum_{k=1}^n \frac{1}{10^k} &= \frac{1}{10^1} - \frac{1}{10^{n+1}} && \text{simplify left side} \\
 \sum_{k=1}^n \frac{1}{10^k} &= \left(\frac{1}{10^1} - \frac{1}{10^{n+1}}\right) \cdot \frac{10}{9} && \text{multiply both sides by } \frac{10}{9} \\
 &= \frac{1}{10^1} \cdot \frac{10}{9} - \frac{1}{10^{n+1}} \cdot \frac{10}{9} && \text{distribute} \\
 &= \frac{1}{9} - \frac{1}{9 \cdot 10^n} && \text{simplify}
 \end{aligned}$$

I rechecked my work. Satisfied, I nodded.

“Now we need to think about what’s going to happen to the right side here as n goes to infinity.”

$$\sum_{k=1}^n \frac{1}{10^k} = \frac{1}{9} - \frac{1}{9 \cdot 10^n}$$

“Hmm,” Tetra said, putting a finger to her lips. “When n goes to infinity, the limit of the $\frac{1}{9 \cdot 10^n}$ part here will be 0, won’t it?”

“It will. That means we can say this.”

$$\sum_{k=1}^n \frac{1}{10^k} \rightarrow \frac{1}{9} \quad \text{as } n \rightarrow \infty$$

“In other words . . .”

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{10^k} = \frac{1}{9}$$

Answer 5-3 (basic limits)

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{10^k} = \frac{1}{9}$$

“You guys done?”

I glanced back and saw Miruka standing behind us, holding some sheet music. I nodded.

“Calculate 0.999 . . . next, then.”

Problem 5-4

Calculate 0.999 . . . , defining 0.999 . . . as follows:

$$0.999 \dots = \lim_{n \rightarrow \infty} 0.\underbrace{999 \dots 9}_n$$

I blinked at this unexpected problem, but soon laughed.

“So *this* is what you’ve been leading us to.”

“You finally noticed,” she said, taking the notebook and solving the problem herself.

$$\begin{aligned}
0.999\dots &= \lim_{n \rightarrow \infty} 0.\underbrace{999\dots 9}_{n \text{ 9's}} \\
&= \lim_{n \rightarrow \infty} \left(0.9 + 0.09 + 0.009 + \dots + 0.\underbrace{000\dots 0 9}_{(n-1) \text{ 0's}} \right) \\
&= \lim_{n \rightarrow \infty} \left(\frac{9}{10^1} + \frac{9}{10^2} + \frac{9}{10^3} + \dots + \frac{9}{10^n} \right) \\
&= \lim_{n \rightarrow \infty} 9 \cdot \left(\frac{1}{10^1} + \frac{1}{10^2} + \frac{1}{10^3} + \dots + \frac{1}{10^n} \right) \\
&= \lim_{n \rightarrow \infty} 9 \cdot \sum_{k=1}^n \frac{1}{10^k} \\
&= 9 \cdot \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{10^k} \\
&= 9 \cdot \frac{1}{9} \quad \text{from Answer 5-3}
\end{aligned}$$

“And thus, $0.999\dots$ equals 1,” Miruka said.

Answer 5-4

$$0.999\dots = \lim_{n \rightarrow \infty} 0.\underbrace{999\dots 9}_{n \text{ 9s}} = 1$$

“Wow,” Tetra said. “So you can just, like, calculate $0.999\dots$ out?”

“Yeah,” I said, “if you’re clever about how you define things.”

“Infinity fools the senses,” Miruka said. “If you try to rely on common sense when you deal with infinity, you’ll get tripped up every time. Not all of us can be an Euler.”

“I see,” Tetra said.

“So don’t rely on your senses, rely on—” Miruka looked at me.

“Logic,” I said.

Miruka turned to Tetra.

"Don't rely on words, rely on—"

"Equations."

Miruka smiled, and Tetra raised her hand.

"And that's why we use this \lim operator, instead of words like 'gets closer and closer to' and all, right?"

Miruka gave a reluctant nod.

"Yes, but the way we've been treating limits so far, the \lim operator isn't much better than just a word."

"Why not?"

"Because we haven't defined it. Not mathematically, at least."

Miruka began walking slowly around us. "At some point we're going to have to leave the words behind."

"But . . . but how?" Tetra asked.

"By using equations instead, of course," Miruka replied.

"You can use equations to *define* limits? Not just find them?"

"Now we can," Miruka replied with a grin. "But actually that's a relatively new development. Cauchy first brought rigorous concepts of limits into mathematics in the early 1800s, but it wasn't until late in that century that Weierstrass finally gave us a full definition using equations."

The door creaked as Ay-Ay reentered the room.

"Break's over!" she announced. "Miruka! Back at it!"

Tetra was still mumbling to herself, a perplexed look on her face.

"Limits . . . ? With equations . . . ?"

Miruka playfully bopped Tetra's head as she headed back toward the piano.

"We'll get there soon," she said, smiling. "To the realm of epsilon-delta."

Index

\wedge (AND), 72
 \cap (intersection), 72
 \vee (OR), 72
 \cup (union), 72
 (ϵ, δ) -definition of a limit, 175
 $x * y$, 343
 $\langle x \rangle$, 343
 $x[n]$, 342
 \bar{n} , 346

A

and(a, b), 353
arbitrarily close, 93, 119
arithmetic, 227, 231
axiom, 27, 156, 224
axiom schema, 156

B

bijection, 75
bound variable, 229

C

CanDivide(x, d), 339
CanDivideByPower(x, n, k),
342

CanDivideByPrime(x, p), 340
Cantor, Georg, 79
Cantor's diagonal argument,
206, 209, 210
 \cap (intersection), 61
cardinality, 79
Cauchy, Augustin-Louis, 110,
136
congruent, 296
ConseqAt(x, n), 358
continuity, 192, 193
contradiction, 70, 310
converging sequence, 98, 124
 $\cos \theta$, 291
countable set, 206
 \cup (union), 63

D

De Morgan's laws, 73
decidable statement, 360
Dedekind, Richard, 79
derivative, 192
diagonalization, 239, 363
diverging sequence, 100, 126

E

element, 28
 empty set, 58
 ϵ -neighborhood, 181
 $\text{equiv}(a, b)$, 353
 equivalence class, 269
 equivalence relation, 267, 296
 equivalence rule, 267
 Euler, Leonhard, 109, 135
 $\exists x \leq M$, 338, 353
 \exists (exists), 178
 extensional definition, 68,
 322, 323

F

factorial(n), 330, 341
 $\forall x \leq M$, 338
 \forall (for all), 31, 178
 $\text{forall}(x, a)$, 345
 formal system

- complete, 229, 310
- incomplete, 229
- of arithmetic, 224
- of propositional logic, 150

 formulas, 319
 free variable, 229, 310
 $\text{freenum}(v, x)$, 352
 $\text{freepos}(k, v, x)$, 352

G

Galilei, Galileo, 78
 Gödel number, 233, 326
 Gödel's incompleteness theorems, 305

H

Hilbert, David, 157, 230, 305,
 306

Hilbert's Program, 308

Hilbert's program, 305

I

immediate consequence, 323,
 357
 $\text{implies}(a, b)$, 353
 \in (element of), 28, 57
 inclusion axiom, 322
 inference rule, 159, 224, 323
 injective mapping, 76
 integral, 192
 intensional definition, 69, 254,
 322
 intersection, 61, 62
 $\text{IsAxiom}(x)$, 357
 $\text{IsAxiomAt}(x, n)$, 358
 $\text{IsAxiomI}(x)$, 354
 $\text{IsAxiomII}(x)$, 355
 $\text{IsAxiomIII}(x)$, 357
 $\text{IsAxiomIV}(x)$, 356
 $\text{IsAxiomV}(x)$, 357
 $\text{IsBoundAt}(v, n, x)$, 350
 $\text{IsConseq}(x, a, b)$, 357
 $\text{IsElementForm}(x)$, 348
 $\text{IsEndedWith}(n, x)$, 350
 $\text{IsForallOp}(x, a)$, 348
 $\text{IsForm}(x)$, 350
 $\text{IsFormSeq}(x)$, 349
 $\text{IsFree}(v, x)$, 351
 $\text{IsFreeAt}(v, n, x)$, 351
 $\text{IsNotBoundIn}(z, y, v)$, 355
 $\text{IsNotOp}(x, a)$, 348
 $\text{IsNthType}(x, n)$, 347
 $\text{IsNumberType}(x)$, 347
 $\text{IsOp}(x, a, b)$, 348
 $\text{IsOrOp}(x, a, b)$, 348

IsPrime(x), 340
IsProof(x), 358
IsProvable(x), 358, 377
IsSchemaII(1, x), 355
IsSchemaII(2, x), 355
IsSchemaII(3, x), 355
IsSchemaII(4, x), 355
IsSchemaIII(1, x), 356
IsSchemaIII(2, x), 356
IsVar(x), 345
IsVarBase(p), 344
IsVarType(x , n), 344

J

\neg (negation), 72

L

Leibniz, Gottfried Wilhelm,
142
len(x), 342
limit, 98, 124

M

$M_5(n)$, 341
 $M_8(x, y)$, 343
 $M_{23}(x)$, 350
mapping, 75
mathematical induction, 41
meta, 27, 219
metamathematics, 324, 344
model theory, 226
modus ponens, 159, 323

N

$n(\bar{n})$, 362
Newton, Isaac, 142
not(x), 345
numeral, 237, 317

O

or(x, y), 345

P

p_n , 341
pair, 246
paren(x), 344
Peano axioms, 25, 320
Peano, Giuseppe, 26
predicate, 41, 229, 336
predicate logic axiom, 321
prime(n, x), 340
prime exponentiation, 329,
342
prime number, 225, 327, 340
primitive recursive predicate,
333
proof, 161, 224
proof by contradiction, 210
proper subset, 66
proposition, 27, 229, 336
propositional axioms, 320
propositional formula, 151,
224, 336
univariate, 238
Proves(p, x), 358, 360
Pythagorean theorem, 283

Q

$Q(x, y)$, 360, 361
quotient set, 269, 297

R

radian, 292
reflexive relation, 266
representative, 271, 297
Russel's paradox, 375

S

scope, 178, 350
self-reference, 375
semantics, 150, 226
 $\sin \theta$, 280, 291
statement, 229, 310, 336
subscript, 97, 123
 \subset (subset), 65
 $\text{subst}(a, v, c)$, 353
 $\text{substAtWith}(x, n, c)$, 351
 $\text{substSome}(k, x, v, c)$, 352
 $\text{succ}(n, x)$, 346
successor, 25, 29, 316, 346
surjective mapping, 76
symmetric relation, 266
syntax, 150

T

theorem, 161, 224
transitive relation, 267
twin prime, 11
 $\text{typelift}(n, x)$, 353

U

undecidable statement, 229,
367
union, 63
unique factorization theorem,
329

V

Venn diagram, 62, 63

W

Weierstrass, Karl, 110, 136,
199, 203
well-defined, 272

Other works by Hiroshi Yuki

(in English)

- *Math Girls*, Bento Books, 2011
- *Math Girls 2: Fermat's Last Theorem*, Bento Books, 2012
- *Math Girls Manga*, Bento Books, 2013
- *Math Girls Talk About Equations & Graphs*, Bento Books, 2014
- *Math Girls Talk About the Integers*, Bento Books, 2014
- *Math Girls Talk About Trigonometry*, Bento Books, 2014

(in Japanese)

- *The Essence of C Programming*, Softbank, 1993 (revised 1996)
- *C Programming Lessons, Introduction*, Softbank, 1994 (Second edition, 1998)
- *C Programming Lessons, Grammar*, Softbank, 1995
- *An Introduction to CGI with Perl, Basics*, Softbank Publishing, 1998

- *An Introduction to CGI with Perl, Applications*, Softbank Publishing, 1998
- *Java Programming Lessons (Vols. I & II)*, Softbank Publishing, 1999 (revised 2003)
- *Perl Programming Lessons, Basics*, Softbank Publishing, 2001
- *Learning Design Patterns with Java*, Softbank Publishing, 2001 (revised and expanded, 2004)
- *Learning Design Patterns with Java, Multithreading Edition*, Softbank Publishing, 2002
- *Hiroshi Yuki's Perl Quizzes*, Softbank Publishing, 2002
- *Introduction to Cryptography Technology*, Softbank Publishing, 2003
- *Hiroshi Yuki's Introduction to Wikis*, Impress, 2004
- *Math for Programmers*, Softbank Publishing, 2005
- *Java Programming Lessons, Revised and Expanded (Vols. I & II)*, Softbank Creative, 2005
- *Learning Design Patterns with Java, Multithreading Edition, Revised Second Edition*, Softbank Creative, 2006
- *Revised C Programming Lessons, Introduction*, Softbank Creative, 2006
- *Revised C Programming Lessons, Grammar*, Softbank Creative, 2006
- *Revised Perl Programming Lessons, Basics*, Softbank Creative, 2006
- *Introduction to Refactoring with Java*, Softbank Creative, 2007

- *Math Girls / Fermat's Last Theorem*, Softbank Creative, 2008
- *Revised Introduction to Cryptography Technology*, Softbank Creative, 2008
- *Math Girls Comic (Vols. I & II)*, Media Factory, 2009
- *Math Girls / Gödel's Incompleteness Theorems*, Softbank Creative, 2009
- *Math Girls / Randomized Algorithms*, Softbank Creative, 2011
- *Math Girls / Galois Theory*, Softbank Creative, 2012
- *Java Programming Lessons, Third Edition (Vols. I & II)*, Softbank Creative, 2012
- *Etiquette in Writing Mathematical Statements: Fundamentals*, Chikuma Shobo, 2013
- *Math Girls Secret Notebook / Equations & Graphs*, Softbank Creative, 2013
- *Math Girls Secret Notebook / Let's Play with the Integers*, Softbank Creative, 2013
- *The Birth of Math Girls*, Softbank Creative, 2013
- *Math Girls Secret Notebook / Round Trigonometric Functions*, Softbank Creative, 2014
- *Math Girls Secret Notebook / Plaza of Sequences*, Softbank Creative, 2014