Math Girls^2

Fermat's Last Theorem

$$a^{\heartsuit} + b^{\heartsuit} = c^{\heartsuit}$$

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Math Girls^2:
Fermat's Last Theorem

Contents

To my readers

This book contains math problems covering a wide range of difficulty. Some will be approachable by middle school students, while others may prove challenging even at the college level.

The characters often use words and diagrams to express their thoughts, but in some places equations tell the tale. If you find yourself faced with math you don't understand, feel free to skip over it and continue on with the story. Tetra and Yuri will be there to keep you company.

If you have some skill at mathematics, then please follow not only the story, but also the math. You might be surprised at what you discover. You may even learn something about the wonderful tale that you yourself are living.

—Hiroshi Yuki

Prologue

God made integers.
All else is the work of man.

LEOPOLD KRONECKER

We count in a world of integers. We count birds, stars, the number of days until the weekend. When we're children, we count to see how long we can hold our breath.

We draw in a world of figures. We use compasses to draw arcs, rulers to draw lines, and we are amazed at the constructions that result. We run through the schoolyard dragging an umbrella, and turn to see a long, winding line stretching toward the horizon.

We live in a world of mathematics. God made the integers, Kronecker said. But Pythagoras and Diophantus bound the integers in right triangles. And then came Fermat... Ah, Fermat, and his silly little note. A problem that anyone could understand, but no one could solve. History's greatest puzzle—if it's fair to call a problem that took mathematicians three centuries to solve a mere "puzzle."

But true forms are hidden. Things once lost are found again. That which has vanished reappears. Loss and rediscovery, death and rebirth. The joy of life and the burden of time.

Consider the meaning of growth.

Question the meaning of solitude.

Know the meaning of words.

Memories are ghosts lost in the mist. I recall only fragments: the silver Milky Way, a warm hand, a trembling voice, chestnut hair.

So that's where I'll begin, on a Saturday afternoon—

Chapter 1

Infinity in Your Hand

Gauss's path was the way of mathematics, a road paved by induction. "From the specific to the general!" was his slogan.

Teiji Takagi
Historical Tales from Modern Mathematics

1.1 The Milky Way

"It's beautiful!" Yuri said.

"Yeah, more like jewels than stars."

I was in the eleventh grade. Yuri was in eighth. She's my cousin on my mother's side, but she hung around my house often enough that people mistook her for my sister. We'd been playmates since we were little, and her house was just down the street.

On days when we didn't have school, it wasn't unusual to find her lounging in my room with her nose buried in one of my books. Today, it was a coffee table book of astronomy.

Vega, Altair, Deneb...

Procyon, Sirius, Betelgeuse...

At one level, a field of stars is nothing but a collection of points of light, but there was something about the fleeting patterns that enthralled us both.

"I heard somewhere that there are two kinds of people," Yuri said. "Those who look up at the night sky and try to count the stars, and those who look for shapes in them. Which are you?"

"The counting kind," I replied. "Definitely the counting kind."

1.2 Discoveries

"So what's high school like?" Yuri asked, her chestnut ponytail bouncing as she replaced the book on my shelf. "Hard?"

"Nah, not really," I said, cleaning my glasses.

"These books sure *look* hard."

"Those aren't my school books. Those are for fun."

"Wait, the *hard* books are the ones you read for *fun*?"

"Learning isn't fun if you don't test your limits."

Her eyes ran along the shelf. "So many math books." She stood on her toes to read the spines of some of the ones higher up.

"Not a math fan?" I asked.

Yuri glanced back at me.

"I don't hate it, but...I definitely don't like it as much as you."

"Marathon math sessions in the library after school aren't for everybody."

"The library? Really?"

"Don't hate on the library. It's cool in the summer, warm in the winter. Plenty to read. Stick me in a library with a math book, a notebook, and a pencil, and I'm pretty much set."

"Let me get this straight. You do math. For fun. And it's not even, like, extra credit or anything?"

"I don't really need extra credit in math."

"And you do...what? Solve for x?"

"Sometimes. When there's an x to solve for. I also mess with equations, draw graphs, you name it."

"Do I dare ask why?"

"I dunno. The beauty of it, I guess."

"Beauty. In *math*." Yuri raised an eyebrow.

"You'd be surprised."

"Okay," she said. "Surprise me."

1.3 Odd One Out

I pulled out my notebook and waved Yuri over to my desk. She dragged a chair up beside mine and took a pair of plastic glasses from her shirt pocket before peering down at the open page.

"You write like a girl," she said.

"That's not my handwriting. It's a quiz a friend wrote for me."

"Fine, your friend writes like a girl."

"I'll be sure to tell her."

Which number doesn't belong?

101	321	681
991	450	811

"Doesn't look like any quiz I've ever taken," Yuri said.

"It's more like a game. Just figure out which number is different from the others."

"No sweat. 450, right?"

"Good. And why is that the odd one out?"

"Because it's the *even* one out. It ends with 0. All the others end in 1."

"Exactly. Okay, how about this one?"

Which number doesn't belong?

11	31	41
51	61	71

"Huh. All of them end in 1 this time."

"You're looking for something else here. This one's still on the easy side, by the way."

"Says you." Yuri crossed her arms. "I give up."

"The answer is 51."

"What's so special about 51?"

"It's the only one that isn't a prime number. You can write 51 as $51 = 3 \times 17$, so it's a composite. You can't do that to the others."

"Somehow I don't feel bad for not knowing that."

"Give the next one a shot:"

Which number doesn't belong?

100	225	121
256	288	361

"256," Yuri said, "because it doesn't have a pair. See how 100 has that 00, and 225 has a 22, and 288 has an 88?"

"What about 121?"

"Still has a pair. Two 1s."

"Okay, then how do you explain away 361?"

"That's the exception that proves the rule?"

"Nice try. The answer's actually 288."

"How come?"

"It's the only one that isn't the square of an integer:"

$$
\begin{array}{lll}
100 = 10^2 & 225 = 15^2 & 121 = 11^2 \\
256 = 16^2 & 288 = 17^2 - 1 & 361 = 19^2
\end{array}
$$

"Again, not knowing proves I'm normal."

"How about a real challenge? This one took me a whole day:"

Which number doesn't belong?

239	251	257
263	271	283

"Much more interesting than this problem is the fact that you could spend a whole day thinking about it."

My mother entered the room with two cups of hot chocolate.

Yuri beamed. "Thanks!"

"How's your foot?" Mom asked.

"It's fine."

"You hurt your foot?" I asked.

"Nothing big," Yuri shrugged. "Sometimes I get these pains in my heel."

"Growing pains, maybe?" my mother suggested.

"Dunno. I'm going to the doctor next week to get it looked at."

"Well, I hope it's nothing serious," she said, scanning my bookshelf. "You're over here so much, you should bring some of your own books. Something more...interesting."

"Oh, I love the books here!" Yuri said. "*And* the hot chocolate!"

"You staying for dinner?"

"Sure! If you don't mind."

"You're always welcome, you know that. Anything in particular you kids want?" She looked back and forth between us.

"Something healthy," Yuri said.

"But tasty," I added.

"And exotic!"

"But...with a Japanese flair."

"Why can't you ask for macaroni and cheese like normal kids," Mother sighed, heading back downstairs. "I'll see what I can whip up."

1.4 Clock Math

"I was promised beauty," Yuri said. "These quizzes aren't cutting it."

"Okay, how about some clock math?"

"Clock math," she repeated, brimming with unenthusiasm.

"Draw a circle." I pushed the notebook and pencil toward her. She sat, staring at it. "You do know what a circle is, right?"

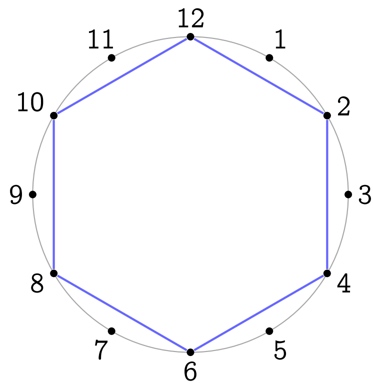
"Gee, I think so." Yuri rolled her eyes and drew.

"Okay, now pretending that's a clock, start from the 12 o'clock position and draw a straight line to the 2 o'clock position, then to 4 o'clock, 6 o'clock, and so on, skipping every other number. Make sense?"

"Sure."

"What happens when you do that?"

Yuri drew the lines.



2 steps

"I got a sixagon, and I ended up back at 12 o'clock."

"A hexagon—you hit 2, 4, 6, 8, 10, and 12, and skip 1, 3, 5, 7, 9, and 11."

"Yeah," Yuri nodded, "hit all the evens, skip all the odds."

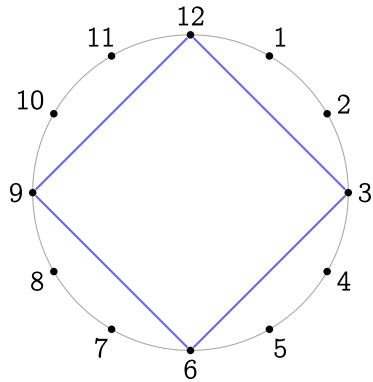
"Good," I said, "you know about evens and odds."

Yuri punched my shoulder. "Just because I'm not a math geek doesn't mean I'm an idiot."

"The jury's still out on that one."

Yuri wound up to throw another punch, so I put jokes aside and returned to the math.

"Okay, let's start a new clock. This time, try connecting every third number, $3, 6, 9$, and 12."



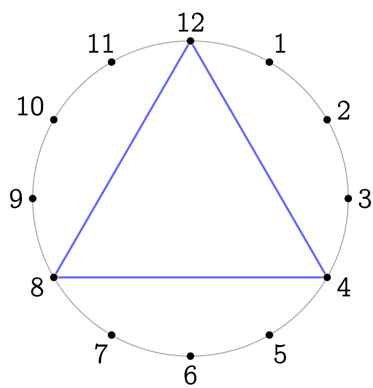
3 steps

"I got a diamond."

"Bravo. Next we're going to make the number of steps 4."

"The number of steps?"

"When we connect every fourth number, I'm going to call that setting the number of steps to 4. So what happens?"

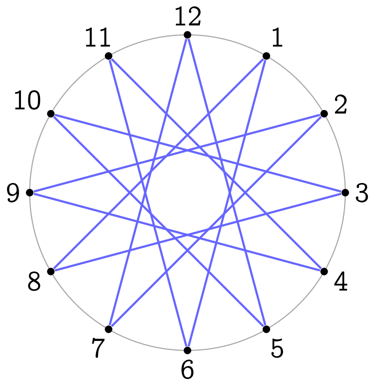


4 steps

"I connected 4, 8, and 12, and got a triangle."

"Okay, here's where it gets good. Next, try connecting every fifth number. In other words—"

"—in other words, the number of steps is 5. I get it, I get it."



5 steps

"Oh, wow. Totally did not see that coming. I hit all of them."

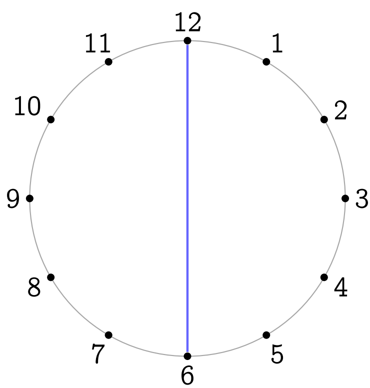
"Right. You completely cycled through the numbers."

"After a few times around the loop, yeah. You miss the 12 a couple times, which lets you hit all the numbers before you get back where you started."

"Let's call doing that 'making a complete cycle.' So moving around the clock when the number of steps is 5 makes a complete cycle."

"Okay."

"So what happens with 6 steps?"

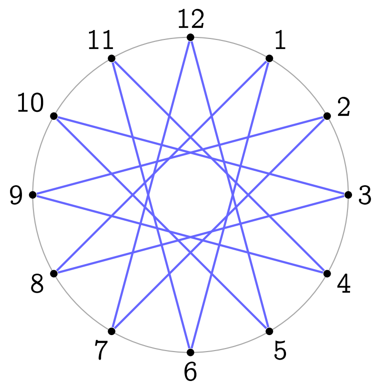


6 steps

"Boringest. Drawing. Ever."

"Maybe we'll hit pay dirt with lucky 7."

"Let's see... 12, then 7, then 2, 9, 4... Looking good."



7 steps

"A complete cycle!" Yuri practically squealed.

"Have you noticed something?"

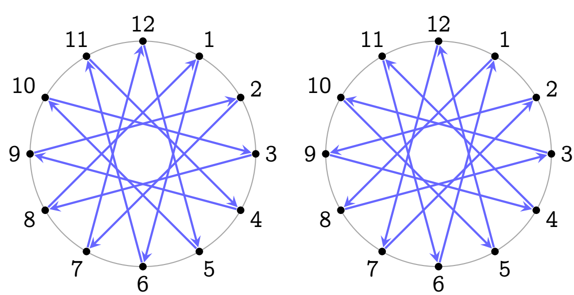
"Like what?"

"Like anything."

Yuri stared at the graph. She pushed her glasses back and tugged on her ponytail.

"What am I not seeing here?"

"Compare it with the 5-step clock. Trace your finger along the lines."

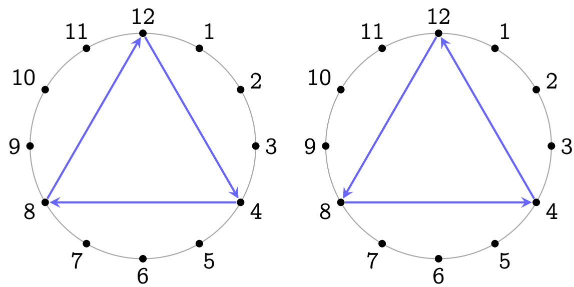


5 and 7 steps

"Hey, they're reversed. Going 7 steps is just like going 5, but backwards."

"Let's see how 8 steps turns out..." I said, reaching for the pencil. Yuri batted me away.

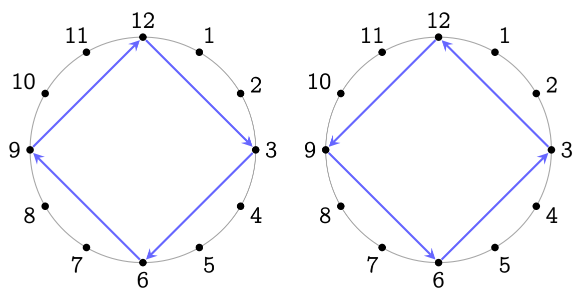
"Hands off."



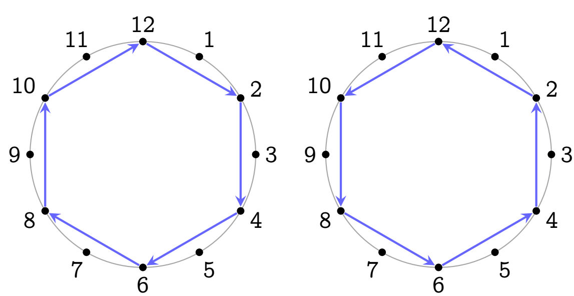
4 and 8 steps

"Cool," Yuri said. "8 is the reverse of 4."

Yuri hurried through a couple more graphs.



3 and 9 steps

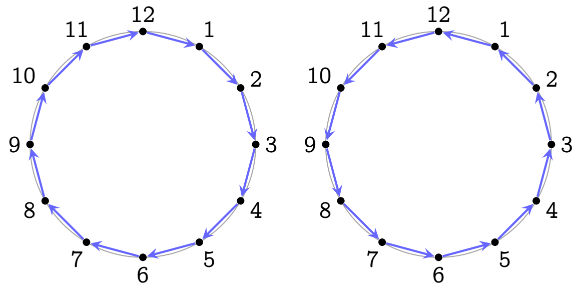


2 and 10 steps

"Y'know, this is kinda neat."

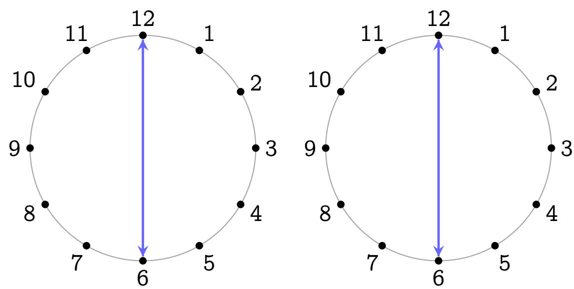
"Don't forget 1 and 11."

"1 step? Oh, you just don't skip anything. That's a complete cycle, too, I guess."



1 and 11 steps

"Poor 6 gets paired with itself."



6 and 6 steps

"And that's all of them," Yuri said. "Huh. Who knew you could learn so much just drawing clocks?"

1.5 Conditions for a Complete Cycle

"So this is what you do in the library?" Yuri asked.

"I do all kinds of things. I was about your age when I first played around with this. I filled a whole notebook with clocks."

"Well then there's got to be more to it than just drawing lines."

"Sure there is. Like, when do you get a complete cycle?"

"We already know that. With $1, 5, 7$, or 11 steps."

"Yeah, but why *those* numbers? Let's write down what we know so far:"

Steps and complete cycles
$1, 5, 7$, and 11 steps result in a complete cycle. $2, 3, 4, 6, 8, 9$, and 10 steps do not result in a complete cycle.

"I just said that."

"Sometimes it helps to write down everything you know. We want to look at these steps and see if we can't find a pattern. Using what you know to find a general rule is called induction. So what do you think determines if you'll get a complete cycle?"

I wrote a problem out in the notebook:

Problem 1-1 (Requirements for a complete cycle)
What property must a number of steps have to result in a complete cycle?

"I have no idea, but this is still kinda cool." She leaned in to whisper, "It's almost like I'm doing real math!"

1.6 Going Cycling

"Let's make a table and list the numbers we cycled through for each number of steps," I said. "We don't care what order we hit them, though:"

1	1	2	3	4	5	6	7	8	9	10	11	12
2	2	4	6	8	10	12						
3	3	6	9	12								
4	4	8	12									
5	1	2	3	4	5	6	7	8	9	10	11	12
6	6	12										
7	1	2	3	4	5	6	7	8	9	10	11	12
8	4	8	12									
9	3	6	9	12								
10	2	4	6	8	10	12						
11	1	2	3	4	5	6	7	8	9	10	11	12

"How do you read this?" Yuri asked.

"The column on the left is the number of steps. Everything on the right is the numbers you cycle through, smallest to largest." I pointed at the third row. "So this row says that with 3 steps, we make four stops, at $3, 6, 9$, and 12."

"Okay."

"Does the table tell you anything?"

"Something about multiples?"

"What about them?"

"Mmm, never mind."

"No, go ahead. What did you notice?"

"Well, it looks like every row is a list of multiples of the first number in the row."

"For example?"

"Like, in the second row. $2, 4, 6, 8, 10$, and 12 are all multiples of 2. And in the row you just talked about, $3, 6, 9$, and 12 are all multiples of 3. So that means to hit all the numbers—to make a complete cycle—the smallest number has to be 1, like for step numbers $1, 5, 7$, and 11. Because 1 is the only number that's a factor of all the numbers!"

"You're absolutely right," I said. "The rows for the number of steps that make a complete cycle start with 1, and *only* those rows do."

"So I solved the problem?"

"Not quite. The problem asks what properties those numbers of steps have. So you have to figure out what kind of steps will have a 1 in the list of numbers that they cycle through."

"Not sure I get it."

"Okay, let's call 'the smallest number you cycle through' the 'minimum cycle number.' You just said a step will give a complete cycle if its minimum cycle number's 1, right?"

"Right."

"So we want to know if there's some way to use the number of steps to figure out the minimum cycle number. Here, I'll write a list of the steps and the minimum cycle numbers you found:"

Steps		Minimum cycle number
1	$\longrightarrow$	1
2	$\longrightarrow$	2
3	$\longrightarrow$	3
4	$\longrightarrow$	4
5	$\longrightarrow$	1
6	$\longrightarrow$	6
7	$\longrightarrow$	1
8	$\longrightarrow$	4
9	$\longrightarrow$	3
10	$\longrightarrow$	2
11	$\longrightarrow$	1

"Do you see a way to go from one to the other?"

"Not really. It starts off nice—$1, 2, 3, 4$—but then it jumps back down to 1."

"Here's a hint. There are twelve positions on the clock, 1 through 12, right? Let's add that to the list:"

Number of values and steps		Minimum cycle number
12 and 1	$\longrightarrow$	1
12 and 2	$\longrightarrow$	2
12 and 3	$\longrightarrow$	3
12 and 4	$\longrightarrow$	4
12 and 5	$\longrightarrow$	1
12 and 6	$\longrightarrow$	6
12 and 7	$\longrightarrow$	1
12 and 8	$\longrightarrow$	4
12 and 9	$\longrightarrow$	3
12 and 10	$\longrightarrow$	2
12 and 11	$\longrightarrow$	1

Yuri coiled her ponytail around her finger while she thought.

"Multiples again? The numbers on the left are all multiples of the number on the right. Like, the fourth one from the bottom. There's a 12 and an 8 on the left, and a 4 on the right. 12 and 8 are both multiples of 4."

"And what does that tell you?"

"I've seen this at school... It's a common multiple! No, wait—the other one—a common divisor! The minimum cycle number on the right is a divisor of the two numbers on the left. Since it's a divisor of both of them, it's a *common* divisor, right? The minimum cycle number is a common divisor of the number of values and the steps!"

"Nice! But you left out one important detail."

"What detail? Oh, wait." She held out a hand to stop me from saying the answer. "I get it. It's not just a common divisor, it's the *greatest* common divisor."

"So when will you get a complete cycle?"

"When the greatest common divisor is 1."

"And *there's* the answer to the problem."

"Woo!"

Answer 1-1 (Requirements for a complete cycle)
A complete cycle occurs when the greatest common divisor (GCD) of the number of values and the steps is 1.

"In other words," I said, "when they're relatively prime."

"What's that mean?"

"What you just said—that their greatest common divisor is 1. Here's a formal definition:"

Relatively prime (coprime)
Two positive integers a and b are called "relatively prime" (or "coprime") if their greatest common divisor (GCD) is 1.

"So 12 and 7 are relatively prime," I said, "since their greatest common divisor is 1, but 12 and 8 aren't, because theirs is 4. So here's another way to answer the problem:"

Answer 1-1a (Requirements for a complete cycle)
A complete cycle occurs when the number of values and the steps are relatively prime.

"You learn something every day," Yuri said.

"Good for you, keeping up like that. Making me explain how to read the table and all. It's important to be sure you understand every step in a problem."

"Nah, I'm just dumb, so I have to ask a lot of questions."

"There's nothing dumb about asking questions. What's dumb is *not* asking them when you don't understand something."

Yuri grinned. "I think that's the first time I've been complimented for not understanding something."

1.7 Beyond Human Limits

"So, all this clock stuff... Is this really math?" Yuri asked.

"Why wouldn't it be?"

"Well, jumping around on clock faces, making tables... It feels more like a game. More than anything I've ever done in class, at least. I mean, just what *is* math?"

"Tough question. I guess math is a lot of things, but investigating the properties of numbers is a big part of it. That branch of mathematics is called 'number theory.' Drawing pictures, making tables, guessing how numbers will behave—it might seem like playing around, but this is really important stuff. Big truths usually aren't easy to see at first, so using tools like induction to go from the specific to the general is key."

"Interesting, maybe. But important? I've gotten by fine without it so far."

"Look at it this way: A normal clock only has twelve positions, so we were able to test all the possible steps ourselves. But what if you wanted to know what happens on a clock with a hundred positions? And if you were patient enough to figure that out by hand, what about one with a thousand? Or a million?"

"Your hand'd probably fall off."

"If you drew them all, sure. But therein lies the true power of mathematics—once you notice that the greatest common divisor of the number of values and the steps tells you everything you need, you don't have to draw anything. Just find the hidden pattern, and you can ride it to places you'd never get to on your own.

"Mathematics is a gateway. It lets you travel though time in a heartbeat. It lets you fold up infinities and hold them in the palm of your hand. That's what's so amazing about it."

"Not as amazing as how worked up you get talking about this stuff. You make my math teacher look like she's allergic to algebra." Yuri laughed. "Speaking of which, you'd make a good math teacher. Bet I'd get better grades in *your* class."

"You'll probably already be graduated by the time I'm old enough to be your teacher... Probably."

"Hey!"

1.8 What Things Really Are

"So I never got the answer to that problem that took you all day," Yuri said. She turned back to the page in my notebook:

Which number doesn't belong?

239	251	257
263	271	283

"Oh, that. It's actually pretty simple, once you hear the answer. All these numbers are primes, which means they're all odd numbers, since 2 is the only even prime. Do you see how dividing an odd number by 2 will always leave a remainder of 1?"

"Yeah, sure."

"The trick here is to divide by 4, not 2. I'll write it out:"

$$239 = 4 \times 59 + 3 \quad 251 = 4 \times 62 + 3 \quad 257 = 4 \times 64 + 1$$

$$263 = 4 \times 65 + 3 \quad 271 = 4 \times 67 + 3 \quad 283 = 4 \times 70 + 3$$

"Um...so what's the answer?"

"When you divide 257 by 4, you get a remainder of 1. For all the others, you get a remainder of 3."

"Okaaay... And why did you try dividing them by 4, exactly?"

"Well, when you're playing with integers, dividing them by 2 is a pretty common trick to check if they're even or odd, since the remainder tells you. Dividing by 4 does something similar, because it leaves a remainder of 1 or 3 for an odd number. Took me a whole day to think of that, though. I was crushed."

"Who wouldn't be," Yuri deadpanned. "Y'know, as nerdy as you can be, you're fun. I enjoyed that clock stuff. Let's do this again sometime." She paused in thought for a moment. "Hey, I have an idea. Why don't you teach me math? So I don't have to drop out of junior high and wait for you to become a real teacher."

"I don't mind teaching you, but only if you promise to think hard for yourself, too. Don't just assume you know things, make *sure* you know them."

"Ha! You sound just like the cat teacher."

"Sorry? Cat teacher?"

"It's this old cartoon movie my dad has. What was it he said... Something like, 'can any of you tell me what that fuzzy white streak *really* is?' "

"Now you're really losing me."

"He was talking about the Milky Way. About how some people used to think it was a river, but really it's millions of tiny stars. The cat teacher asks this kid about it, but the kid doesn't know. Turns out the cat teacher doesn't know either, not really. The kid rides on the Milky Way Railroad, and finds out the truth."

"Oh, 'Night on the Galactic Railroad.' I've heard of that."

"That's the one!"

" 'Can you tell me what it *really* is,' huh? I like that," I said. "Always a good thing to ask."

My mother called from the kitchen. "Dinnertime! Come on down for some healthy, tasty, exotic, Japanese eggplant curry!"

Chapter **2**

Pythagorean Triples

> Then along comes the Taniyama-Shimura conjecture, the grand surmise that there's a bridge between these two completely different worlds. Mathematicians love to build bridges.
>
> Simon Singh
> *Fermat's Enigma*

2.1 Rooftop Lunch

"You okay?" Tetra asked.

"Wait...what?" I blinked, my eyes regaining their focus.

"Sorry," Tetra grinned. "Didn't mean to wake you."

Tetra and I had gone to the roof to eat lunch. There was a bite to the wind, but it felt good to be out under blue skies. Tetra was working at a bento box with a pair of chopsticks, while I picked at a muffin.

"Guess I kinda zoned there."

She smiled. "You had me worried for a minute."

Tetra was in her first year of high school, one year behind me. She was a small girl with short hair, big eyes, and a smile that rarely left her face. She was one of my "math friends." *I* was supposed to be tutoring *her*, but her unique take on things often meant our roles were reversed.

"Hey, did you get a card from Mr. Muraki?" I asked.

Mr. Muraki was our math teacher. He had taken a liking to us, and would regularly slip us index cards with all sorts of interesting math problems. They rarely had anything to do with our classwork, which made for a refreshing change of pace. We always looked forward to what he would come up with next.

"Oh, yeah! Completely forgot!" Tetra took out a small card and handed it to me. I read it at a glance; it was just a single line:

Problem 2-1
Are there infinitely many primitive Pythagorean triples?

"That's it?"

"Guess so," Tetra mumbled around a mouthful of fried egg.

"So you know about Pythagorean triples."

"Well, duh! Who doesn't?" Tetra traced a right triangle in the air with her chopsticks. " 'The square of the hypotenuse is equal to the sum of the squares of the sides,' right?"

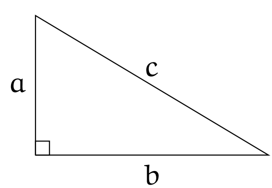
I sighed.

"Er, *not* right?"

"That's the Pythagorean *theorem*." I picked up Tetra's notebook and wrote the full statement:

The Pythagorean theorem

In any right triangle, the square of the length of the hypotenuse c is equal to the sum of the squares of the lengths of the legs a, b.



$$a^2 + b^2 = c^2$$

"So...they're different."

"Yeah, but now that you mention it, I guess you could think of a Pythagorean triple as the lengths of the sides of a right triangle, only where each length is an integer. Like this:"

Pythagorean triple

Three positive integers a, b, and c are called a "Pythagorean triple" if $a^2 + b^2 = c^2$.

"The definition of a *primitive* Pythagorean triple is just a little bit different:"

Primitive Pythagorean triple

Three *relatively prime* positive integers a, b, and c are called a "primitive Pythagorean triple" if $a^2 + b^2 = c^2$.

"So if you have a right triangle where the length of each side is an integer, then those three lengths form a Pythagorean triple. If the lengths are also relatively prime, then they form a *primitive* Pythagorean triple. Your card is asking if there are an infinite number of those."

"Okay... No, not okay. What does 'relatively prime' mean?"

"That their greatest common divisor is 1."

Tetra raised her eyebrows.

"Here, let me give you an example. $(3, 4, 5)$ is a Pythagorean triple. See?" I wrote in her notebook:

$$3^2 + 4^2 = 5^2$$
$$9 + 16 = 25$$

"$(3, 4, 5)$ is also a *primitive* Pythagorean triple, because the biggest number that evenly divides these numbers—their greatest common divisor—is 1."

"Okay, I'm with you so far. Can you show me a Pythagorean triple that isn't primitive?"

"Sure, just double these three and see what happens."

"So $(6, 8, 10)$? Well, 6^2 is 36, 8^2 is 64, and 10^2 is 100, and $36+64 = 100$, so yeah, this has to be a Pythagorean triple." Tetra paused and rested her chopsticks on her lips. "Oh, but 2 is a factor of 6, 8, and 10, so 1 can't be the greatest common divisor, which means this isn't a *primitive* Pythagorean triple."

"See, that wasn't so bad."

"I'm still missing something, though. Since $a^2 + b^2 = c^2$ will work for any right triangle, and you can just change the lengths of the sides to make an infinite number of right triangles, why *wouldn't* there be an infinite number of primitive Pythagorean triples?"

"You're right. You are missing something. Go back and look at the conditions in the definition."

Tetra stared at the definition. "Ah ha!" She jabbed the notebook with a chopstick. "Here's what I forgot. They all have to be integers. It's easy to set up a right triangle so that *two* of the sides have an integer length, but that doesn't mean the *third* side will."

"Exactly! The way to start working on this problem is to search for other Pythagorean triples like $(3, 4, 5)$, and see if you notice anything. Remember—"

"—'Examples are the key to understanding,' right?" She poked me in the chest with a chopstick. "I wondered how long it'd take to get to that."

2.2 Rational Points

Miruka pounced as soon as I walked into homeroom.

"And just where have you been?" she demanded.

Miruka was smart in general, but pure genius when it came to math. Her hair was long and black, and she shunned contacts in favor of a simple pair of metal frame glasses. She was tall, and beautiful, and I felt an almost electric charge in the air just being around her.

"Um...on the roof?"

"Lunch?"

"Yeah. Lunch."

She leaned in, peering into my eyes to see what truths might be hiding there. I detected a hint of citrus in the air.

"And you didn't invite me?"

"Yeah. Uh, no. I mean, you weren't around. I figured you were off with Ay-Ay or something..."

How did I end up on the defensive here?

"I was in the teachers' office, handing in a report to Muraki. I guess he was expecting me; he already had a new problem waiting." Miruka handed me her card. "Weird one this time."

Problem 2-2

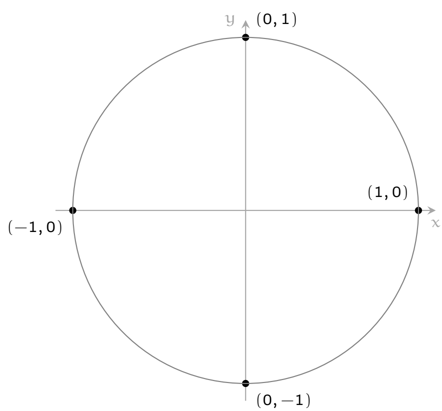
Are there infinitely many rational points on the unit circle?

"I guess a rational point is one where the x- and y-coordinates are both rational numbers? Numbers that can be represented as a ratio of integers, like $\frac{1}{2}$ or $-\frac{2}{5}$?"

Miruka nodded. "Obviously there are at least *some* rational points on the unit circle—$(1, 0)$ and $(0, -1)$, for example."

"Sure, the intersections with the x- and y-axes. That makes sense."

"But I'm not sure there are an infinite number of them."



The unit circle and four obvious rational points

"You'd think there'd be an infinite number of others, though," I said, half to myself.

"What makes you say that?"

"Well, there are an awful lot of rational points lying around. Seems like it would be hard to draw a circle that missed them all."

" 'Seems like' won't cut it in math," Miruka declared. "Never trust your intuition when you have the tools to be precise."

"Fine," I rolled my eyes, "let's do this the right way then. Take integers a, b, c, d and represent a point as $(\frac{a}{b}, \frac{c}{d})$, establish the condition that the point has to be on the unit circle, and start crunching away."

"Hmm. That's one way to do it." Miruka turned back to the circle on the chalkboard. "Prime factorization reveals the integers. Ratios of integers reveal the rationals." There was a music in her voice now. "I wonder if there isn't some way to set up a correlation between rational points on the unit circle and an infinite number of *somethings...*"

The corners of her mouth turned up in a mischievous smile.

"But what I'm really wondering is..." she began.

"Is what?"

"Is who you were having lunch with."

"Oh." *Just when I thought I was safe.* "Tetra."

Miruka stared at me for a moment, her expression impenetrable. Finally she straightened, taking a step away from my desk. "For your honesty and bravery, sir knight, I bestow upon you this sword," Miruka said, brandishing a Kit Kat. I took it from her just as the bell rang for class.

Rational points, I could handle. *Girls, I will* never *understand.*

2.3 Yuri

The next day after school I took a bus straight downtown to the hospital, Tetra in tow. The doctors had found something wrong when they checked out Yuri's foot.

We arrived to find Yuri sitting up in bed reading a book, her hair tied back with a yellow ribbon.

"Oh, you didn't have to come visit," she said, but her smile made it plain how happy she was to see me.

"How you feeling?"

"I'm fine. I don't know why everyone is making such a fuss." Her smile faded when she noticed Tetra. "Who's that?"

"A friend from school," I said.

Tetra held out a small bouquet of flowers she'd bought on the way to the hospital. "I'm Tetra. Nice to meet you." Yuri accepted the flowers without a word.

I sat down in the chair beside Yuri's bed. Tetra found another chair and busied herself looking around the room.

"Thanks again for showing me that clock math the other day," Yuri said. "It was really cool how you could make it around all the numbers if everything was relatively prime."

"He's a great teacher, isn't he?" Tetra said. "He's showing me—"

"And that curry!" Yuri interrupted. "It was *sooo* spicy. I drank way too much water. Oh, and that stuff about Fermat's last theorem you told me about after dinner—that was way cool, too."

Tetra shifted in her seat.

Yuri's mother appeared in the doorway, breaking the uncomfortable silence. "Look how handsome you are in your uniform. That was sweet, coming straight from school. And this must be your *girlfriend*!" The conversation only went downhill from there, and after a few minutes that felt like hours, Tetra and I politely excused ourselves. We were almost to the elevator when Yuri's mother caught up with us.

"You don't mind if I borrow your girlfriend for a minute, do you? Yuri wants to tell her something."

2.4 The Pythagorean Juicer

We took the bus back to the train station and decided to drop into Beans before heading home.

"You were pretty quiet on the ride back," I said. "What did Yuri want?"

"Nothing major," Tetra replied, before spotting the perfect conversation changer. "Oh, cool! Look!"

Beans had installed a fruit juicer. Wire rails supporting a neat queue of oranges spiraled from its mouth. The barrista hit a button and the next condemned fruit dropped into the machine, where it was guillotined and squeezed, its juices streaming into the glass waiting below.

"Oh, I have *got* to have one of those," Tetra said, squealing with glee as another orange plummeted to its doom.

* * *

"*So* good," Tetra said, setting down her glass. She flipped her notebook open and slid it across the table. "Look what I brought—more Pythagorean triples!"

$$\begin{array}{ll} (3,\ 4,\ 5) & 3^2 + 4^2 = 5^2 \\ (5,12,13) & 5^2 + 12^2 = 13^2 \\ (7,24,25) & 7^2 + 24^2 = 25^2 \\ (8,15,17) & 8^2 + 15^2 = 17^2 \\ (9,40,41) & 9^2 + 40^2 = 41^2 \end{array}$$

"How'd you find these?"

"Well, I started with $a^2 + b^2 = c^2$, and increased a one at a time. Then I just played around with b and c, setting them to whatever. I noticed something interesting—in four of the triples I found, c is just one more than b. Doesn't that look like some kind of clue?"

"That probably has a lot to do with how you're searching. When a is small, you're going to have a triangle with one short side, right? Like with $(9, 40, 41)$ there, you get this long, thin triangle. That means the length of the hypotenuse is going to be close to the length of the third side."

"Oh yeah, I guess you're right." Tetra cocked her head. "Too bad we don't have a machine that spits them out for us. A Pythagorean juicer! Just dump a bunch of triangles in, and wait for it to squeeze out a bucket full of primitive Pythagorean triples!"

I rested my face in my hands. "That would make this problem easier, no doubt."

2.5 Primitive Pythagorean Triples Revisited

I loved nighttime. With my family in bed and the world asleep, there were no distractions, no interruptions. I was free to sit at my desk, thinking about math. This was *my* time.

Class can provide some interesting insights, and there's much to be learned from books, but without some quality time putting pencil to paper, it's never enough.

That night, Tetra's problem kept bouncing around in my head: *Are* there an infinite number of primitive Pythagorean triples?

2.5.1 Checking Parity

I started with the list Tetra had made:

a	b	c
3	4	5
5	12	13
7	24	25
8	15	17
9	40	41

I noticed c was odd every time. Intrigued, I circled the odd numbers in the table:

a	b	c
(3)	4	(5)
(5)	12	(13)
(7)	24	(25)
8	(15)	(17)
(9)	40	(41)

Interesting. Exactly one of a *or* b *is odd in every case. Coincidence? Or something deeper...?*

I jotted down a self-posed problem:

Problem 2-3

Does there exist a primitive Pythagorean triple (a, b, c) where a and b are both even?

The answer came surprisingly quickly.

Nope, not gonna happen.

Since (a, b, c) was a Pythagorean triple, I knew that $a^2 + b^2 = c^2$. If a and b were both even, that meant a^2 and b^2 would be even too. Their sum, $a^2 + b^2$, would also be even, as would c^2, since the two were equal. But only even numbers can be squared to produce another even number, which meant c also had to be even.

So if you start with an even a and b, you'll always get an even c. But (a, b, c) is a primitive Pythagorean triple, which means those numbers are relatively prime. And if all three are even, their greatest common divisor will be at least 2—a contradiction! It was impossible for a and b to both be even.

I wasn't sure if that would help with Tetra's problem, but it was an interesting find nonetheless. Whenever I wandered through the forest of mathematics and stumbled across something like this, I made a point of writing it down on a ribbon and tying it to a nearby branch. When you get lost, it's good to have a way to retrace your steps.

Answer 2-3
There does not exist a primitive Pythagorean triple (a, b, c) such that a and b are both even.

2.5.2 Trying Equations

Now I knew a and b in a primitive Pythagorean triple couldn't both be even. But could they both be *odd*?

Problem 2-4
Does there exist a primitive Pythagorean triple (a, b, c) where a and b are both odd?

I decided to try the same approach as before, assuming a and b were odd and seeing what fell out.

If a was odd, then a^2 would be odd, too. Same with b and b^2. Then $a^2 + b^2$ would be a sum of two odd numbers, which would be even, meaning c^2 had to be even since $a^2 + b^2 = c^2$. If c^2 is even then c must be even, meaning it's a multiple of 2. Since c^2 was the product of two multiples of two, it in turn must be a multiple of 4.

I sat back and stared at what I'd scribbled down, wondering if this was getting me anywhere. Not seeing any obvious next steps, I decided to try building some equations and see where that led.

My premise was that a and b were both odd. That meant I could use two new positive integer variables J and K, and write a and b like this:

$$\begin{cases} a &= 2J - 1 \\ b &= 2K - 1 \end{cases}$$

I also knew these variables had to work with the definition of Pythagorean triples, which gave me my starting point:

$$\begin{aligned}
a^2 + b^2 &= c^2 && \text{definition of a Pythagorean triple} \\
(2J-1)^2 + (2K-1)^2 &= c^2 && \text{from } a = 2J-1, b = 2K-1 \\
(4J^2 - 4J + 1) + (4K^2 - 4K + 1) &= c^2 && \text{expand} \\
4J^2 - 4J + 4K^2 - 4K + 2 &= c^2 && \text{clean up} \\
4(J^2 - J + K^2 - K) + 2 &= c^2 && \text{factor out a 4}
\end{aligned}$$

The $+2$ dangling at the end of the left side of the equation put a smile on my face—that was a remainder when dividing by 4, meaning the left side wasn't a multiple of 4. But I'd just found that c^2 *had* to be a multiple of 4. And that meant... *Contradiction.*

So my premise that a and b are both odd must be false, proving they could *not* both be odd.

Answer 2-4
There does not exist a primitive Pythagorean triple (a, b, c) such that a and b are both odd.

Combining this with my previous proof, I'd shown that one of a or b had to be even. In other words, a and b had to have different parity.

That meant there were two possible cases: either a was odd and b was even, or a was even and b was odd. I decided to let a be the odd one, and b the even one. Anything I proved under that assumption could always be proved for the other case, just by swapping the letters.

My stomach growled as my pencil headed back towards my notebook.

2.5.3 As a Product

I pulled the Kit Kat from Miruka out of my bag, recalling something she had said that day. "Prime factorization reveals the integers..."

I definitely wanted to get a deeper look into $a^2 + b^2 = c^2$, but how to apply prime factorization to it? Even if I couldn't manage a product of primes, *some* kind of product might be helpful:

$$
\begin{aligned}
a^2 + b^2 &= c^2 && \text{definition of a Pythagorean triple} \\
b^2 &= c^2 - a^2 && \text{move } a^2 \text{ to create a difference of squares} \\
b^2 &= (c+a)(c-a) && \text{a difference of squares is the product of a sum and a difference}
\end{aligned}
$$

I had my product now, but it wasn't doing me any good. I couldn't claim that $c + a$ and $c - a$ are primes, so I was a long way from revealing anything.

Then I realized I was having a Tetra moment—I had forgotten the conditions of the problem, despite all that time I spent figuring out the parity of a and b. Since I said a is odd and b is even, c must be odd. And if c and a are both odd, that means $c + a$ and $c - a$ must be even:

$$
\begin{aligned}
\text{odd} + \text{odd} &= \text{even} \\
\text{odd} - \text{odd} &= \text{even}
\end{aligned}
$$

Since c and a were both odd, I had this:

$$
\begin{aligned}
c + a &= \text{even} \\
c - a &= \text{even}
\end{aligned}
$$

I introduced three new positive integers A, B, and C, and used them to set up equations for $c + a$, $c - a$, and b, the things that I now knew were even:

$$\begin{cases} c - a &= 2A \\ b &= 2B \\ c + a &= 2C \end{cases}$$

I worried A might end up a negative number, but soon realized that couldn't happen, since a, b, and c were the lengths of sides of a right triangle, with c the hypotenuse. A hypotenuse would always be the longest side, guaranteeing that $c > a$, and thus that $2A > 0$.

Okay, let's play around with A, B, *and* C *then.*

$a^2 + b^2 = c^2$	definition of a Pythagorean triple
$b^2 = c^2 - a^2$	move a^2 to create a difference of squares
$b^2 = (c + a)(c - a)$	a difference of squares is the product of a sum and a difference
$(2B)^2 = (2C)(2A)$	substitute $A, B,$ and C
$4B^2 = 4AC$	multiply
$B^2 = AC$	divide both sides by 4

Now I had the definition of Pythagorean triples converted into a product, with a little help from the positive integers A, B, and C.

I'd gotten a lot of mileage out of investigating the parity of a, b, and c, but I was still wandering in the woods without a path in sight.

A square on the left, a product on the right. Which way to head next?

2.5.4 Relatively Prime

I stared at $B^2 = AC$, wondering what it was trying to tell me. Finally I stood up and started walking in circles to clear my head.

On my fifth trip past the bookshelf, I recalled something I had told Yuri when she was browsing my books.

Sometimes it's good to summarize what you know.

I went back to my desk and wrote down a list:

- $c - a = 2Ab = 2B$
- $c + a = 2C$
- $B^2 = AC$
- a and c are relatively prime

The last item gave me pause. *Do I really know that?*

I had included it because the definition of primitive Pythagorean triples says that a, b, and c are relatively prime. But just because the greatest common divisor of *all three* numbers is 1, that doesn't necessarily mean it would be the case for any *two* of them. For example, 1 is the greatest common divisor of 3, 6, and 7, but if you just looked at 3 and 6, the GCD would be 3.

After some thought I convinced myself I was safe in the case of a primitive Pythagorean triple, thanks to the equation $a^2 + b^2 = c^2$. Here's how I proved it:

> For a primitive Pythagorean triple (a, b, c), assume that the GCD of a and c is some number g greater than 1. Then there exist positive integers J, K such that $a = gJ, c = gK$. Then b^2 is a multiple of g^2, as follows:
>
> $$\begin{aligned} a^2 + b^2 &= c^2 \\ b^2 &= c^2 - a^2 \\ b^2 &= (gK)^2 - (gJ)^2 \\ b^2 &= g^2(K^2 - J^2) \end{aligned}$$
>
> b is therefore a multiple of g, as are a and c. However this contradicts the definition of a primitive Pythagorean triple, which states that a, b, and c are relatively prime. The assumption that the GCD of a and c is greater than 1 must therefore be false, meaning their GCD must be 1, and thus that a and c are relatively prime. A similar argument can be used to show that (a, b) and (b, c) are likewise relatively prime.

Working that out was a relief, but then I started wondering about A and C. Could they be relatively prime too?

Problem 2-5

For relatively prime a and c where $c-a=2A, c+a=2C$, are A and C relatively prime?

My gut told me they were, but as Miruka would no doubt be happy to remind me, that wasn't good enough. I had to prove it.

Proof by contradiction had brought me this far, so I decided to stick with it. In this case, the statement I wanted to prove was "A and C are relatively prime," so I would start from the assumption that they *aren't*. That meant their greatest common divisor was greater than 1. I decided to again call the greatest common divisor 'g.'

I knew that $g \geqslant 2$, and since g was the greatest common divisor of A and C it was a divisor of each. Looked at another way, A and C were multiples of g. That meant there existed positive integers A' and C' that satisfied this:

$$\begin{cases} A = gA' \\ C = gC' \end{cases}$$

From the problem, I had this:

$$\begin{cases} c - a = 2A \\ c + a = 2C \end{cases}$$

I tried writing a and c in terms of A' and C', starting with c:

$$\begin{aligned}
(c+a)+(c-a) &= 2C+2A && \text{add the equations} \\
2c &= 2(C+A) && \text{clean up both sides} \\
c &= C+A && \text{divide both sides by 2} \\
c &= gC'+gA' && A, C \text{ in terms of } A', C' \\
c &= g(C'+A') && \text{factor out } g
\end{aligned}$$

The last line told me that c was a multiple of g.

Next, I tried finding an expression for a:

$$
\begin{aligned}
(c+a)-(c-a) &= 2C-2A && \text{subtract the equations}\\
2a &= 2(C-A) && \text{clean up both sides}\\
a &= C-A && \text{divide both sides by 2}\\
a &= gC'-dA' && A, C \text{ in terms of } A', C'\\
a &= g(C'-A') && \text{factor out } g
\end{aligned}
$$

From this I learned that a was a multiple of g, too. So my $g \geqslant 2$ was a common divisor of both a and c. But I'd started out saying a and c were relatively prime, which meant their greatest common divisor had to be 1. The contradiction had to be a result of my premise that A and C were *not* relatively prime, so using proof by contradiction I'd shown that they are.

Answer 2-5
For relatively prime a and c where $c-a=2A, c+a=2C$, A and C are relatively prime.

So now I had established that A and C are relatively prime. Another ribbon, though I still wasn't sure if it would be useful. I sat back and took a deep breath. My eyelids were getting a little heavy, but the forest beckoned.

2.5.5 Prime Factorization

Looking back through my notes, the equation $B^2 = AC$ caught my eye as a square that was also the product of two relatively prime numbers.

Interesting...

Problem 2-6

- A, B, C are positive integers.
- $B^2 = AC$.
- A and C are relatively prime.

Find something interesting.

Not my finest problem statement.

Ah, well. It's late.

I realized I'd left the a, b, c of my original problem behind, and was now only dealing with A, B, C. It took me a second to even remember the original problem: Are there an infinite number of primitive Pythagorean triples? I hoped I wasn't going too far astray.

Miruka's unusual comment kept popping into my head. "Prime factorization reveals the integers."

I wondered what a prime factorization of A, B, and C would look like.

Something like this, I guess:

$$\begin{aligned} A &= a_1 a_2 \cdots a_s && a_1 \text{ through } a_s \text{ are prime} \\ B &= b_1 b_2 \cdots b_t && b_1 \text{ through } b_t \text{ are prime} \\ C &= c_1 c_2 \cdots c_u && c_1 \text{ through } c_u \text{ are prime} \end{aligned}$$

I tried combining that with $B^2 = AC$ to see what would happen:

$$\begin{aligned} B^2 &= AC && \text{given} \\ (b_1 b_2 \cdots b_t)^2 &= (a_1 a_2 \cdots a_s)(c_1 c_2 \cdots c_u) && \text{prime factorization} \\ b_1^2 b_2^2 \cdots b_t^2 &= (a_1 a_2 \cdots a_s)(c_1 c_2 \cdots c_u) && \text{expand the left side} \end{aligned}$$

Hmmm...

Writing out a prime factorization of B^2 puts it in a form where every prime factor b_k gets squared, meaning there would be an even number of each factor. Using 18^2 as an example, you end up with $18^2 = (2 \times 3 \times 3)^2 = 2^2 \times 3^4$, two 2s and four 3s.

From the uniqueness of prime factorization, which says there's only one way to do the prime factorization of any integer, I knew the factors on the left and right sides of this equation would have to match perfectly; every factor that showed up on the left side would have to be on the right side somewhere, and vice versa.

Out of the corner of my eye, I caught a glimpse of my second ribbon, the one that said A and C are relatively prime, fluttering in a chance breeze.

That's it! A and C can't have any prime factors in common!

A prime factor b_k of B couldn't be a factor of both A and C. Using $2^2 \times 3^4$ as an example again, I thought about how that number could be represented as a product of positive integers A and C.

If there's even one 2 in the prime factorization of A, then all of 2^2 had to be there. And if there's even one 3 in the prime factorization of A, then all of 3^4 had to be there. There would never be a case where a prime factor of B^2 was split up between A and C. That meant the factorization of $2^2 \times 3^4$ could only be one of the following four cases:

A	C
1	$2^2 \times 3^4$
2^2	3^4
3^4	2^2
$2^2 \times 3^4$	1

Since there had to be an even number of each prime factor, *A and C both have to be square numbers!*

Answer 2-6

If

- A, B, C are positive integers,
- $B^2 = AC$, and
- A and C are relatively prime,

then A and C are square numbers.

Alright, now that's just cool.

Since A and C are squares, I could represent them in terms of positive integers m and n:

$$\begin{cases} C &= m^2 \\ A &= n^2 \end{cases}$$

Introducing more variables made me cringe, but I was pretty sure I'd glimpsed a path out of this thicket, so I forged ahead. If I got lost, I could just follow the trail back through my notes.

Since A and C don't have any prime factors in common, m and n are of course relatively prime. So now I should be able to write a, b, c in terms of relatively prime numbers m, n.

Starting with $a = C - A$, I could say this:

$$a = C - A = m^2 - n^2$$

Since $a > 0$, I knew that $m > n$. I'd also said a is odd, which meant m and n had to have different parity.

Next, because $c = C + A$, I knew this had to be true:

$$c = C + A = m^2 + n^2$$

The last piece in the puzzle was $b = 2B$.

This one will take a little fiddling.

$$\begin{aligned} B^2 &= AC \\ B^2 &= (n^2)(m^2) && \text{from } A = n^2, C = m^2 \\ B^2 &= (mn)^2 && \text{cleaning up} \\ B &= mn && \text{safe to take root, because } B > 0, mn > 0 \end{aligned}$$

From this, I got:

$$b = 2B = 2mn$$

Finally, I could write a, b, c in terms of relatively prime numbers m, n:

$$(a, b, c) = (m^2 - n^2,\ 2mn,\ m^2 + n^2)$$

I could also go the other way, starting with m and n and using this to create a primitive Pythagorean triple. I tried it out, just for kicks:

$$\begin{aligned}
a^2 + b^2 &= \left(m^2 - n^2\right)^2 + b^2 && \text{from } a = m^2 - n^2 \\
&= \left(m^2 - n^2\right)^2 + \left(2mn\right)^2 && \text{from } b = 2mn \\
&= m^4 - 2m^2n^2 + n^4 + 4m^2n^2 && \text{expand} \\
&= m^4 + 2m^2n^2 + n^4 && \text{combine the } m^2n^2 \text{ terms} \\
&= (m^2 + n^2)^2 && \text{factor} \\
&= c^2 && \text{use } c = m^2 + n^2
\end{aligned}$$

Some simple calculations would also be enough to show that a, b, c were relatively prime.

Examining parity, paying attention to conditions of relative primeness, and prime factorization had yielded a wonderful treasure—a general form for primitive Pythagorean triples. It had taken some doing to find them, but find them I had.

A general form for primitive Pythagorean triples

All relatively prime positive integers (a, b, c) such that $a^2 + b^2 = c^2$ can be written in the following form (note that a and b can be interchanged):

$$\begin{cases} a = m^2 - n^2 \\ b = 2mn \\ c = m^2 + n^2 \end{cases}$$

where

- m, n are relatively prime,
- $m > n$, and
- one of m, n is even, the other odd.

From here, Tetra's problem practically solved itself. Since different primes would of course be relatively prime, you could just use an array of them to generate an infinite number of primitive Pythagorean triples. For example, using $n = 2$ and $m = 3$ as a starting point, just advance m through $3, 5, 7, 11, 13, \cdots$, and each distinct pair of m, n would give you a new (a, b, c).

I was exhausted, but that night I fell asleep with a smile on my face.

Answer 2-1
There are infinitely many primitive Pythagorean triples.

2.6 Finding the Trail

I met Tetra in the library the next day and showed her my solution—a Pythagorean juicer that would spit out a primitive triple every time you dropped in an m and an n.

"Oh, come on!" She practically shouted. "He expected me to do *that*?"

"Shhh!"

"Sorry, sorry." She lowered her voice. "Look, this is cool and all, but it's way over me. I'm never going to be able to sit down and come up with something like this off the top of my head."

"Nobody can. You just have to trudge along until you stumble across what you're looking for. Tell you what, how about I walk you through it, show you why I did what I did."

"Sure, I guess."

I squared my shoulders. "First off," I said, "the fact that we're working with integers, not real numbers, is a big deal here. Well only positive integers, strictly speaking. But anyway, the important thing is that there's no smooth continuity like with the reals. The integers are discrete. Lumpy.

"When you're dealing with integers, thinking about parity—whether a number is even or odd—can tell you a lot. It *only* helps with integers, though, since the reals don't have evenness or oddness."

I grabbed Tetra's notebook and wrote "parity" in it.

"When you have an equation in the form ⟨integer⟩ = ⟨integer⟩, you know both sides of the equation will have the same parity. It's also helpful to remember that adding two odd numbers gives an even number, and so does multiplying any number by an even number."

I added "prime factorization" to the burgeoning list in Tetra's notebook.

"You can also learn a lot from prime factorization. Factoring an integer into primes pulls it apart so you can see what it's built from. Also, if you have an equation in the form ⟨integer⟩ = ⟨integer⟩, the uniqueness of prime factorization tells you the factorization of both sides will be the same. That can be useful too."

"How do I do that?"

"Start by putting things in multiplicative form. The numbers that make up the product are called factors. Like in the AC product we were talking about, the A and the C are factors."

"Okay. How does having the factors help?"

"Well, you see how a single prime can't be split across multiple factors, right, because a prime can't be broken down any further. So if you have a product of two factors, a single prime factor will have to be completely contained in one of them. That's why I used the 'product of a sum and a difference is the square of a difference' rule to write this here as a product of two integers."

I pointed to the step in my notebook.

"Knowing how to write words as mathematical symbols is important, too. Like writing 'even number' as $2k$, or 'odd number' as $2k - 1$, or 'square number' as k^2. It takes some getting used to, but it's not hard. Remember how you compared writing math to writing an essay? Well, you can think of $2k - 1$ representing an odd number as a common mathematical shorthand."

"Oh, I like that," Tetra said, adding "mathematical shorthand" to the list herself. Beneath that, I wrote "relative primeness."

"Relative primeness is another important concept. If you know two numbers are relatively prime, you know they don't share any prime factors, which was key to solving this problem."

"At the end here, right? Gotcha."

"So you just chip away at the problem with a bunch of different tools. You keep looking until you find the trail leading out of the forest."

Tetra let out a long sigh.

"It's a lot to take in," I said.

"Yeah, but I'm getting there. One thing, though. You know how you kept making new variables, like an 'odd number' variable and a 'square number' variable and all? I'm *really* not good at that. Too afraid it's just going to make the problem harder, I guess."

"I know what you mean. Things can get messy if you don't, though."

"If you say so..." Tetra skimmed back over what I'd written. "So when I'm dealing with integers, I should check parity, try prime factorization, put them in multiplicative form, divide numbers by their greatest common divisor to make them relatively prime..."

"Well, those are all things to try, but no guarantee they'll lead to anything."

"I know, they're just ways to look for the trail."

"Right. And if you get lost, you can always backtrack and look for something else."

2.7 Squeaky

"This was a really interesting problem," I said. "Feels like there's still more to be found, deeper in. Something about the nature of numbers..."

"I know I've said this before," Tetra said, her voice somber, "but I want to thank you."

"For what?" I said.

"Showing me things. I worked hard on this problem. I really did! But you've shown me things that would have taken me forever to find, if I ever found them at all. The parity and prime factorization stuff is part of it, but it's more than that. I'm starting to get a feel for what it's like doing math with integers. They're kinda, I don't know, *squeaky*."

We both laughed.

"But seriously, I think I underestimated them. For some reason I thought they'd be easier to deal with than real numbers. But they aren't. They're just...different."

Tetra's cheeks flushed.

"When I'm talking with you, I always learn something different than what I get from class, or books. I thought I already knew all this stuff. Pythagorean theorem? Got it! Integers? No problem! But it turns out I hardly knew the first thing..." She shook her head. "No giving up now! I might still be deep in the forest, but I've got a good guide."

2.8 Rational Points on the Unit Circle

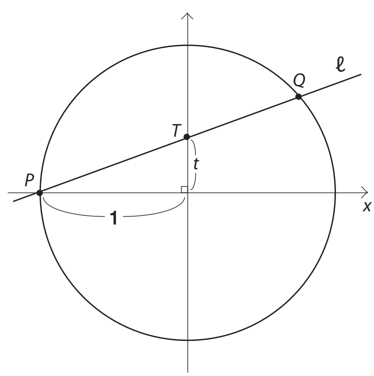
The next day, Miruka and I were hanging out in our classroom after school. She had promised me a fun proof that there are an infinite number of rational points on the unit circle.

"It all depended on finding an infinite number of *somethings*," she said. "Everything falls into place after that."

She went to the board and picked up a piece of chalk. My eyes followed her hand as it traced out a large circle with uncanny precision.

"First, let's review the problem," she said. "We begin with a point (x, y) and a circle with radius 1, centered at the origin—the unit circle, which is defined as $x^2 + y^2 = 1$. The problem asks if there are an infinite number of rational points on the circle. In other words, if there are an infinite number of rational solutions to $x^2 + y^2 = 1$.

"Start off drawing a line ℓ passing though point $P(-1, 0)$ with slope t:"



"Since ℓ's slope is t, it passes through $T(0, t)$, and its equation is:"

$$y = tx + t$$

"If we ignore the case where ℓ is tangent to the circle at P, we know it has to intersect the circle at some point other than P. Let's call that point Q. We can figure out Q's coordinates in terms of t by solving this system of equations, since the solution is their intersection:"

$$\begin{cases} x^2 + y^2 = 1 & \text{equation for the circle} \\ y = tx + t & \text{equation for } \ell \end{cases}$$

"Okay, let's solve it:"

$$\begin{aligned} x^2 + y^2 &= 1 && \text{equation for the circle} \\ x^2 + (tx + t)^2 &= 1 && \text{substituted } y = tx + t \\ x^2 + t^2x^2 + 2t^2x + t^2 &= 1 && \text{expanded} \\ x^2 + t^2x^2 + 2t^2x + t^2 - 1 &= 0 && \text{moved the 1} \\ (t^2 + 1)x^2 + 2t^2x + t^2 - 1 &= 0 && \text{factored out } x^2 \end{aligned}$$

"Nice," I said. "We know that $t^2 + 1 \neq 0$, so it's a quadratic equation now. We can just solve it using the quadratic formula."

"We could, but we already know that $x = -1$ is a solution, since it's the x-coordinate of the point $P(-1, 0)$, so what's the point? Let's factor out that solution, an $x + 1$, instead:"

$$(x + 1) \cdot \big((t^2 + 1)x + (t^2 - 1)\big) = 0$$

"In other words, we have this:"

$$x + 1 = 0 \quad \text{or} \quad (t^2 + 1)x + (t^2 - 1) = 0$$

"Now we've got x in terms of t:"

$$x = -1, \quad \frac{1 - t^2}{1 + t^2}$$

"We can also write y in terms of t, using the equation for a line. We know $(x, y) = (-1, 0)$ isn't Q, so we only need to pay attention to $x = \frac{1-t^2}{1+t^2}$:"

$$\begin{aligned} y &= tx + t \\ &= t\left(\frac{1-t^2}{1+t^2}\right) + t \\ &= \frac{t(1-t^2)}{1+t^2} + t \\ &= \frac{t(1-t^2)}{1+t^2} + \frac{t(1+t^2)}{1+t^2} \\ &= \frac{t(1-t^2) + t(1+t^2)}{1+t^2} \\ &= \frac{2t}{1+t^2} \end{aligned}$$

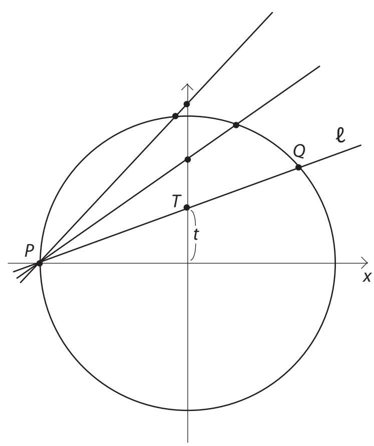
"So now we have the coordinates of Q:"

$$\left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right)$$

"Here's where the cool trick comes. Ready?"

"Fire away."

"Remember this point T on the y-axis? How its y-coordinate is t?"



"Sure."

"See how Q's coordinates are just combinations of basic arithmetic operations on t?"

"Yeah. So?"

" 'So?' If you perform basic arithmetic on a rational number, you're going to get another rational number back. Which means—?"

"—that if you make t a rational number, Q will be a rational point. Right."

"Since there are an infinite number of rational numbers we can let t be, and since every one will result in a different point Q, we have our answer."

Answer 2-2

There are infinitely many rational points on the unit circle.

"Huh, pretty slick," I said.

Miruka looked at me, plainly expecting something.

"What?"

"You still don't see it?"

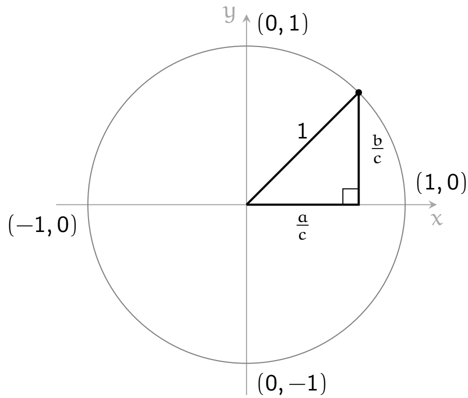
"See what?"

"Wow, you're dense today. *Tetra's card.* Divide the Pythagorean theorem through by c^2. What do you get?"

$$\left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = 1$$

"Oh, cool! $(x, y) = (\frac{a}{c}, \frac{b}{c})$ is a solution to $x^2 + y^2 = 1$. You can squeeze a unit circle out of the Pythagorean theorem with this!"

"The rational points on a unit circle, at least. So there's a different rational point $(\frac{a}{c}, \frac{b}{c})$ for each primitive Pythagorean triple. Saying there are an infinite number of primitive Pythagorean triples and that there are an infinite number of rational points on the unit circle is basically the same thing. The two cards are basically *the same problem!*"



The relation between the unit circle and Pythagorean triples

My jaw dropped.

Miruka shook her head. "You're slipping."

I had seen both cards, but Tetra's card had been all about integers, and Miruka's about rational points. Apparently that's all it took to throw me off—it never occurred to me that the problems might be related.

"Yeah, I shoulda caught that."

"Certainly not the first time Muraki's had something up his sleeve. Finding solutions to an equation is pure algebra. Playing with circles is straight out of geometry." She looked up at me for a long moment before concluding, "I guess he wanted to show us a bridge between the two."

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