## Part (1)

Define sequence $\langle p_n \rangle$ as

$$p_1 = 3, \quad p_{n+1} = \frac{1}{3}p_n + 1 \quad (n = 1, 2, 3, \ldots). \qquad \cdots\cdots\cdots\cdots ①$$

We wish to find a general term for $\langle p_n \rangle$, and the sum of its first $n$ elements. First, from ① we have that

$$p_{n+1} - \frac{\boxed{\text{A}}}{\boxed{\text{B}}} = \frac{1}{3}\left(p_n - \frac{\boxed{\text{A}}}{\boxed{\text{B}}}\right) \quad (n = 1, 2, 3, \ldots),$$

so a general term for $\langle p_n \rangle$ is

$$p_n = \frac{1}{\boxed{\text{C}} \cdot \boxed{\text{D}}^{n-2}} + \frac{\boxed{\text{E}}}{\boxed{\text{F}}}.$$

Therefore, for a natural number $n$,

$$\sum_{k=1}^{n} p_k = \frac{\boxed{\text{G}}}{\boxed{\text{H}}}\left(1 - \frac{1}{\boxed{\text{I}}^n}\right) + \frac{\boxed{\text{J}}n}{\boxed{\text{K}}}.$$

## Part (2)

Let $a_1 = 3, a_2 = 3, a_3 = 3$ be the first three terms in a sequence of positive numbers $\langle a_n \rangle$ such that for all natural numbers $n$

$$a_{n+3} = \frac{a_n + a_{n+1}}{a_{n+2}}. \qquad \cdots\cdots\cdots\cdots ②$$

Furthermore, define sequences $\langle b_n \rangle, \langle c_n \rangle$ such that for all natural numbers $n$, $b_n = a_{2n-1}, c_n = a_{2n}$. We wish to find general terms for sequences $\langle b_n \rangle, \langle c_n \rangle$.

First, from ② we have that

$$a_4 = \frac{a_1 + a_2}{a_3} = \boxed{\text{L}}, a_5 = 3, a_6 = \frac{\boxed{\text{M}}}{\boxed{\text{N}}}, a_7 = 3.$$

From this we obtain $b_1 = b_2 = b_3 = b_4 = 3$, so one might conjecture that

$$b_n = 3 \qquad (n = 1, 2, 3, \ldots). \qquad \cdots\cdots\cdots\cdots ③$$

To demonstrate ③, it is sufficient to show that since $b_1 = 3$, for all natural numbers $n$

$$b_{n+1} = b_n. \qquad \cdots\cdots\cdots\cdots ④$$

We can prove this by showing that ④ holds when $n = 1$, then showing that if ④ holds when $n = k$, then ④ must also hold when $n = k + 1$. Doing a mathematical proof in this manner is called proof by $\boxed{\text{O}}$.

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| What is the most appropriate phrase to insert into $\boxed{\text{O}}$?<br>1. synthetic division 2. circular measure<br>3. mathematical induction 4. contradiction |

**(I)** When $n = 1$, we have $b_1 = 3, b_2 = 3$, and so ④ holds.

**(II)** Suppose that ④ holds when $n = k$. In other words, suppose that

$$b_{k+1} = b_k. \qquad \cdots\cdots\cdots\cdots ⑤$$

Then when $n = k + 1$, we can substitute the $n$ in ② with $2k$ and with $2k - 1$ to obtain

$$b_{k+2} = \frac{c_k + \boxed{\text{P}}_{k+1}}{\boxed{\text{Q}}_{k+1}}, \quad c_{k+1} = \frac{\boxed{\text{R}}_k + c_k}{\boxed{\text{S}}_{k+1}}.$$

Therefore, $b_{k+2}$ can be expressed as

$$b_{k+2} = \frac{\left(\boxed{\text{T}}_k + \boxed{\text{U}}_{k+1}\right)\boxed{\text{V}}_{k+1}}{b_k + c_k}.$$

From ⑤ it thus follows that $b_{k+2} = b_{k+1}$, and so ④ also holds for $n = k + 1$.

From (I), (II) above, we have shown that ④ holds for all natural numbers $n$. Therefore ③ holds, and the general term for sequence $\langle b_n \rangle$ is $b_n = 3$.

Next, from ③ and from substituting the $n$ in ② with $2n - 1$, we obtain

$$c_{n+1} = \frac{1}{3}c_n + 1 \quad (n = 1, 2, 3, \ldots).$$

Since $c_1 = \boxed{\text{W}}$ and from ①, we can see that the general term for the sequence $\langle c_n \rangle$ is equivalent to the general term for sequence $\langle p_n \rangle$ derived in part 1 of this problem.