Math Girls

$$\sum_{k=0}^{\infty} \heartsuit^{k}$$

Hiroshi Yuki

Translated by Tony Gonzalez

Bento Books

http://bentobooks.com

Translated by Tony Gonzalez with Joseph Reeder.
Additional editing by Alexander O. Smith and Brian Chandler.

Published by

Bento Books, Inc.
217 Tierra Grande Court
Austin, TX 78732-2458
USA

bentobooks.com

ISBN 978-0-9839513-1-5 (hardcover)
ISBN 978-0-9839513-0-8 (trade paperback)
Library of Congress Control Number: 2011937467

Printed in the U.S.A.
First edition, November 2011

Math Girls

To My Readers

This book contains math problems covering a wide range of difficulty. Some will be approachable by middle school students, while others may prove challenging even at the college level.

The characters often use words and diagrams to express their thoughts, but in some places equations tell the tale. If you find yourself faced with math you don't understand, feel free to skip over it and continue on with the story. Tetra will be there to keep you company.

If you have some skill at mathematics, then please follow not only the story, but also the math. You might be surprised at what you discover.

—Hiroshi Yuki

Contents

To My Readers **5**

Prologue **1**

1 Sequences and Patterns **3**
- 1.1 Beneath a Cherry Tree 3
- 1.2 Outlier 6
- 1.3 Beginning the Pattern 8

2 Equations and Love Letters **11**
- 2.1 Two Plus One Equals Three 11
- 2.2 Some Mental Arithmetic 12
- 2.3 The Letter 12
- 2.4 Ambush 13
- 2.5 Tetra 14
  - 2.5.1 Defining Prime Numbers 15
  - 2.5.2 Defining Absolute Values 18
- 2.6 Beneath an Umbrella 20
- 2.7 Burning the Midnight Oil 21
- 2.8 Miruka's Answer 24
- 2.9 Math by the Letters 26
  - 2.9.1 Equations and Identities 26
  - 2.9.2 The Forms of Sums and Products 30
- 2.10 Who's Behind the Math? 33

3 Shapes in the Shadows 35
3.1 Rotations 35
3.2 Projections 37
3.3 The Omega Waltz 43

4 Generating Functions 49
4.1 In the Library 49
4.1.1 Searching for Patterns 50
4.1.2 Sums of Geometric Progressions 51
4.1.3 Infinite Geometric Series 52
4.1.4 Creating Generating Functions 53
4.2 Capturing the Fibonacci Sequence 54
4.2.1 The Fibonacci Sequence 54
4.2.2 A Generating Function 56
4.2.3 Searching for a Closed Expression 57
4.2.4 On to the Infinite Series 59
4.2.5 The Solution 60
4.3 Reflections 64

5 Arithmetic and Geometric Means 67
5.1 The Second Note 67
5.2 Questions 68
5.3 Inequalities 70
5.4 Forging Ahead 77
5.5 A Glimpse Behind the Curtain 80

6 The Discrete World 85
6.1 Derivatives 85
6.2 Differences 89
6.3 Bringing Them Together 91
6.3.1 Linear Functions 91
6.3.2 Quadratic Functions 92
6.3.3 Cubic Functions 94
6.3.4 Exponential Functions 95
6.4 Traveling Between Two Worlds 97

7 Generalizations 101
7.1 Back in the Library 101

7.1.1 With Miruka 101
7.1.2 With Tetra 104
7.1.3 Recurrence 105
7.2 Heading Home 108
7.3 The Binomial Theorem 109
7.4 Convolution 116
7.5 Miruka's Solution 122
7.5.1 A Different Approach 122
7.5.2 Confronting Generating Functions 127
7.5.3 The Scarf 130
7.5.4 The Final Hurdle 130
7.5.5 The Leap 133
7.5.6 A Circle with Zero Radius 137

8 Harmonic Numbers 139
8.1 Treasure Hunt 139
8.1.1 Tetra 139
8.1.2 Miruka 140
8.2 For Every Library There Exists a Dialogue... 141
8.2.1 Partial Sums and Infinite Series 141
8.2.2 Starting with the Obvious 144
8.2.3 Propositions 145
8.2.4 For All... 148
8.2.5 There Exists... 150
8.3 An Infinite Upward Spiral 153
8.4 An Ill-Tempered Zeta 155
8.5 Overestimating Infinity 156
8.6 Harmony in the Classrooom 162
8.7 Two Worlds, Four Operators 165
8.8 Keys to the Known, Doors to the Unknown 172
8.9 If There Were Only Two Primes 173
8.9.1 Converging Geometric Progressions 175
8.9.2 The Uniqueness of Prime Factorizations 176
8.9.3 The Infinitude of Primes 176
8.10 Under the Stars 180

9 Taylor Series and the Basel Problem 185

9.1 Two Cards 185
9.1.1 One for Tetra 185
9.1.2 Polynomials of Infinite Degree 187
9.2 Learning on Your Own 190
9.3 Getting Closer 191
9.3.1 Differentiation Rules 191
9.3.2 Looping Derivatives 193
9.3.3 The Taylor Series for $\sin x$ 196
9.3.4 Taking Equations to the Limit 200
9.4 The Problem Within 202
9.5 The Fundamental Theorem of Algebra 205
9.6 Tetra's Attempt 210
9.6.1 Trial and Error 210
9.6.2 A Solution Emerges 212
9.6.3 Defying Infinity 217

10 Partitions 221
10.1 Partition Numbers 221
10.1.1 Ways to Pay 221
10.1.2 Some Concrete Examples 222
10.2 The Fibonacci Sign 228
10.3 Confessions 230
10.4 Making Selections 232
10.5 Three Solutions 235
10.5.1 Using Generating Functions 236
10.5.2 Using an Upper Limit 243
10.5.3 Using Brute Force 248
10.6 Lost and Found 251
10.7 In Search of a Better Upper Bound 252
10.7.1 Starting with Generating Functions 252
10.7.2 Changing Sums into products 254
10.7.3 Taylor Series 255
10.7.4 Harmonic Numbers 259
10.7.5 Coming Home 261
10.7.6 Tetra's Notes 263
10.8 Goodbyes 264

Recommended Reading 273

Index **285**

Other works by Hiroshi Yuki **289**

Prologue

> It is not enough to memorize.
> One must also remember.
>
> HIDEO KOBAYASHI

We met in high school.

I'll never forget them: The brilliant Miruka, forever stunning me with her elegant solutions. The vivacious Tetra, with her earnest stream of questions. It was mathematics that led me to them.

Mathematics is timeless.

When I think back on those days, equations seem to pop into my head and fresh ideas flow like a spring. Equations don't fade with the passage of time. Even today they reveal to us the insights of giants: Euclid, Gauss, Euler.

Mathematics is ageless.

Through equations, I can share the experiences of mathematicians from ages past. They might have worked their proofs hundreds of years ago, but when I trace the path of their logic, the thrill that fills me is mine.

Mathematics leads me into deep forests and reveals hidden treasures. It's a competition of intellect, a thrilling game where finding the most powerful solution to a problem is the goal. It is drama. It is battle.

But math was too hefty a weapon for me in those days. I had only just gotten my hand around its hilt, and I wielded it clumsily—like I handled life, and my feelings for Miruka and Tetra.

It is not enough to memorize. I must also remember.

It all started my first year of high school—

Chapter 1

Sequences and Patterns

One, two, three. One three.
One, two, three. Two threes.

Yumiko Oshima
The Star of Cottonland

1.1 Beneath a Cherry Tree

The entrance ceremony on my first day of high school was held on a fine spring day in April. The principal gave a speech, filled with the usual things people are supposed to say at times like these. I remember maybe half.

...unfolding blossoms that you are... on this occasion of a new beginning... the proud history of this school... excel in your studies, as you excel in sports... learn while you are young...

I pretended to adjust my glasses to hide a yawn.

On my way back to class after the ceremony, I slipped away behind the school, and found myself strolling down a row of cherry trees.

I'm 15 now, I thought. *15, 16, 17, then graduation at 18. One fourth power. One prime.*

$$
\begin{aligned}
15 &= 3 \cdot 5 \\
16 &= 2 \cdot 2 \cdot 2 \cdot 2 = 2^4 && \text{a fourth power} \\
17 &= 17 && \text{a prime} \\
18 &= 2 \cdot 3 \cdot 3 = 2 \cdot 3^2
\end{aligned}
$$

Back in class everyone would be introducing themselves. I hated introductions. What was I supposed to say?

"Hi. My hobbies include math and, uh... Well, mostly just math. Nice to meet you."

Please.

I had resigned myself to the idea that high school would turn out much as middle school had. Three years of patiently sitting through classes. Three years with my equations in a quiet library.

I found myself by a particularly large cherry tree. A girl stood in front of it, admiring the blossoms. Another new student, I assumed, skipping class just like me. My eyes followed her gaze. Above us, the sky was colored in blurred pastels. The wind picked up, enveloping her in a cloud of cherry petals.

She looked at me. She was tall, with long black hair brushed back from metal frame glasses. Her lips were drawn.

"One, one, two, three," she said in precise, clipped tones.

1 1 2 3

She stopped and pointed in my direction, obviously waiting for the next number. I glanced around.

"Who, me?"

She nodded silently, her index finger still extended. I was taken off guard by the pop quiz, but the answer was easy enough.

"The next number is 5. Then 8, and after that 13, and then 21. Next is, uh..."

She raised her palm to stop me and then issued another challenge. "One, four, twenty-seven, two hundred fifty-six."

1 4 27 256

She pointed at me again. I immediately noticed the pattern.

"I guess the next number is 3125. After that... uh, I can't do that in my head."

Her expression darkened. "1—4—27—256—3125—46656," she said, her voice clear and confident. The girl closed her eyes and inclined her head toward the cherry blossoms above us. Her finger twitched, tapping the air.

She was by far the strangest girl I'd ever met, and I couldn't take my eyes off her.

Her gaze met mine. "Six, fifteen, thirty-five, seventy-seven."

6 15 35 77

Four numbers, again, but the pattern wasn't obvious. My head went into overdrive. *6 and 15 are multiples of 3, but 35 isn't. 35 and 77 are multiples of 7...* I wished that I had some paper to write on.

I glanced at the girl. She was still standing at attention beneath the cherry tree. A cherry petal came to rest in her hair, but she didn't brush it away. She didn't move at all. Her solemn expression made our encounter feel all the more like a test.

"Got it."

Her eyes sparkled and she showed a hint of a smile.

"6—15—35—77... and then 133!"

My voice was louder than I'd intended.

She shook her head, sending the cherry petal fluttering softly to the ground. "Check your math," she sighed, a finger touching the frame of her glasses.

"Oh. Oh right. 11×13 is 143, not 133."

She continued with the next problem. "Six, two, eight, two, ten, eighteen."

6 2 8 2 10 18

Six numbers this time. I thought for a bit. The 18 at the end really threw me, since I was expecting a 2. I knew there had to be another pattern, though, and it hit me when I realized that all the numbers were even.

"Next is 4—12—10—6." I frowned. "Kind of a trick question, though."

"You got it, didn't you?"

She approached, her hand extended for an unexpected handshake. I took it, still unsure exactly what was going on. Her hand was soft and warm in mine.

"Miruka," she said. "Nice to meet you."

1.2 Outlier

I loved nighttime. Once my family was in bed, my time was my own. I would spread books before me, and explore their worlds. I would think about math, delving deep into its forests. I would discover fantastic creatures, tranquil lakes, trees that stretched up to the sky.

But that night, I thought about Miruka. I recalled our handshake. The softness of her hand, the way she smelled. She smelled like. . . like a girl. It was clear that she loved math; it was also clear that she was strange—an outlier. Not many people introduced themselves with a pop quiz. I wondered if I had passed.

I laid my glasses on my desk and closed my eyes, reflecting on our conversation.

The first problem, $1, 1, 2, 3, 5, 8, 13, \cdots$, was the Fibonacci sequence. It starts with 1, 1, and each number after that is the sum of the two before it:

$$1, 1, 1+1=2, 1+2=3, 2+3=5, 3+5=8, \cdots$$

The next problem was $1, 4, 27, 256, 3125, 46656, \cdots$. That was a sequence like this:

$$1^1, 2^2, 3^3, 4^4, 5^5, 6^6, \cdots$$

The general term for this would be n^n. I could manage calculating 4^4 and even 5^5 in my head, but 6^6? No way.

The problem after that was $6, 15, 35, 77, 143, \cdots$. I got that by multiplying:

$$2 \times 3, 3 \times 5, 5 \times 7, 7 \times 11, 11 \times 13, \cdots$$

In other words <a prime number> × <the next prime number>. I couldn't believe I'd messed up multiplying 11×13. Check your math, indeed.

The last problem was $6, 2, 8, 2, 10, 18, 4, 12, 10, 6, \cdots$. In other words it was π, but with each digit doubled:

$$\begin{array}{lll} \pi = 3.141592653\cdots & & \textit{pi} \\ \rightarrow & 3, 1, 4, 1, 5, 9, 2, 6, 5, 3, \cdots & \text{digits of } \pi \\ \rightarrow & 6, 2, 8, 2, 10, 18, 4, 12, 10, 6, \cdots & \text{each digit doubled} \end{array}$$

What I didn't like about this problem was that you can't solve it unless you can remember the digits of π. You need to have the pattern already in your head. It relies on memorization.

Math isn't about dredging up half-remembered formulas. It's about making new discoveries. Sure, there are some things that require rote memorization: the names of people and places, words, the symbols of the elements. But math isn't like that. With a math problem, you have a set of rules. You have tools and materials, laid out on the table in front of you. Math's not about *memorizing*, it's about *thinking*. Or at least, that's what it is to me.

I noticed something interesting about that last problem. Miruka had given me six numbers, instead of four like the rest. If she had just said, "$6, 2, 8, 2$," then the series wouldn't necessarily be a doubling of the digits of π. Another, simpler pattern might be possible. Even if she had said, "$6, 2, 8, 2, 10$," I could have answered with a series of even numbers separated by the number 2, like this:

$$6, \underline{2}, 8, \underline{2}, 10, \underline{2}, 12, \underline{2}, \cdots$$

So she knew exactly what she was doing when she made that series longer than the others. And it hadn't surprised her when I got it right. I could still see the smug curl of her mouth.

Miruka.

I remembered the way she looked, standing there in the spring light, her black hair a sharp contrast against the pink petals on the wind, slim fingers moving like a conductor's. The warmth of her hand. Her fragrance.

That night, I couldn't think of much else.

1.3 Beginning the Pattern

By May, the novelty of being in a new school, going to new classrooms, and meeting new friends had largely faded. The days of same old, same old had begun.

I didn't participate in any after-school activities, preferring instead to get away from school stuff as soon as I could. I wasn't particularly good at sports, and just hanging out with friends had never appealed to me. That's not to say I headed straight home. When classes were done, I often went to the school library to do math, a habit I picked up in middle school. No clubs for me; just reading, studying, and staring out the library window.

But my favorite thing to do was tinker with the math I learned in class. Sometimes I would start with definitions, seeing where they led me. I jotted down concrete examples, played with variations on theorems, thought of proofs. I would sit for hours, filling notebook after notebook.

When you're doing math, you're the one holding the pencil, but that doesn't mean you can write just anything. There are rules. And where there are rules, there's a game to play—the same game played by all the great mathematicians of old. All you need is some fresh paper and your mind. I was hooked.

I had assumed it was a game I would always play alone, even in high school. It turned out I was wrong.

Miruka shared homeroom with me, and she showed up in the library a few days a week. The first time she walked up, I was sitting alone, working on a problem. She took the pencil out of my hand,

and started writing in my notebook. In *my* notebook! I wasn't sure if I should be offended or impressed. I decided on the latter.

Her math was hard to follow, but interesting. Exciting, even.

"What was with the sequence quiz the other day?" I asked.

"What other day?" She looked up, hand paused in mid-calculation. A pleasant breeze drifted through the open window, carrying the indistinct sounds of baseball practice and the fragrance of sycamore trees. "I'm drawing a blank here," she said, tapping a pencil—my pencil—against her temple.

"When we first met. You know. . . under the cherry tree."

"Oh, that. They just popped into my head. So what?"

"I don't know. Just wondering."

"You like quizzes?"

"Sure, I guess."

"You guess, huh? Did you know there aren't any right answers to those kinds of sequence problems?"

"What do you mean?"

"Say the problem was $1, 2, 3, 4, \cdots$. What's the answer?"

"Well, 5 of course."

"Not necessarily. What if the numbers jumped after that, say to $10, 20, 30, 40$, and then 100, 200, 300, 400? It's still a perfectly fine sequence."

"Yeah, but, you can't just give four numbers and then say the numbers jump. There's no way to know that a 10 will come after the 4."

"Well, how many numbers do you need, then?" she asked, raising an eyebrow. "If the sequence goes on forever, at what point can you figure out the rest?"

"All right, all right. So there's always a chance that the pattern will suddenly change somewhere beyond what you've seen. Still, saying that a 10 comes after 1, 2, 3, 4 makes for a pretty random problem."

"But that's the way the world works. You never know what's going to come next. Predictions fail. Check this out." Miruka started writing in my notebook. "Can you give me a general term for this sequence?"

$$1, 2, 3, 4, 6, 9, 8, 12, 18, 27, \cdots$$

"Hmmm... Maybe," I said.

"If you only saw the 1, 2, 3, 4, then you'd expect the next number to be 5. But you'd be wrong. Rules won't always reveal themselves in a small sample."

"Okay."

"And if you saw 1, 2, 3, 4, 6, 9, you'd expect the sequence to keep increasing, right? But not here. The number after the 9 is an 8. You see a pattern of increasing numbers, but then *bam*, one goes backwards. Have you found the pattern yet?"

"Well, except for the 1 they're all multiples of 2 or 3. I can't figure out why that one number becomes smaller, though."

"Here's a solution to play with," she said, writing:

$$2^0 3^0,\ 2^1 3^0,\ 2^0 3^1,\ 2^2 3^0,\ 2^1 3^1,\ 2^0 3^2,\ 2^3 3^0,\ 2^2 3^1,\ 2^1 3^2,\ 2^0 3^3,\ \cdots$$

"Look at the exponents on the 2s and 3s. That should help you see it."

"Well, I know that anything raised to the zero power is 1, so yeah, when I do the calculations I get the right sequence," I said, writing underneath:

$$2^0 3^0 = 1,\ \ 2^1 3^0 = 2,\ \ 2^0 3^1 = 3,\ \cdots$$

"But I still don't get it."

"The exponents aren't enough, huh? How about this?"

$$\underbrace{2^0 3^0,}_{\text{sum=0}}\ \underbrace{2^1 3^0,\ 2^0 3^1,}_{\text{sum=1}}\ \underbrace{2^2 3^0,\ 2^1 3^1,\ 2^0 3^2,}_{\text{sum of exponents=2}}\ \underbrace{2^3 3^0,\ 2^2 3^1,\ 2^1 3^2,\ 2^0 3^3,}_{\text{sum of exponents=3}}\ \cdots$$

"Oh, I see it now," I said.

"Hey, speaking of multiples of 2 and 3—" Miruka began, but a shout from the library entrance interrupted her.

"Are you coming, or what?"

"Oh, right. Practice today," Miruka said.

She returned my pencil and headed towards the girl standing in the doorway. On her way out, she looked back. "Remind me to tell you what the world would be like if there were only two prime numbers."

Then she was gone and I was alone again.

Chapter 2

Equations and Love Letters

You fill my heart.

Moto Hagio
Ragini

2.1 Two Plus One Equals Three

My second year of high school was pretty much the same as the first, except that the "I" on my school badge became a "II." Days flowed in an endless stream, each seeming just like the one that had come before, until a cloudy morning in late April.

Less than a month had passed since the start of the new school year. I was walking through the school gates on my way to class when an unfamiliar girl called out to me.

"This is for you," she said, offering me a white envelope. Confused, I took it. She bowed curtly before scurrying off.

I peeked inside—it was a letter. But there was no time to read it. Stuffing the envelope into a pocket, I ran off to class.

It was the first time a girl had given me a letter since elementary school. I'd caught a bad cold and was absent for a couple of days. One of my classmates came by my house to drop off homework, along with a note that said "Get better soon!"

The letter felt heavy in my pocket all during class.

Just like Miruka said, you never know what's going to come next.

2.2 Some Mental Arithmetic

I had just finished eating lunch and was pulling out the girl's letter when Miruka plopped down next to me, nibbling on a candy bar.

"Pop quiz," she announced. "1024. How many divisors?"

"I gotta do it in my head?" I asked, cramming the letter back into my pocket.

"Yep, and before I count to ten. 1, 2, 3—"

I scrambled to think of numbers that would evenly divide 1024. Definitely 1 and 2, but not 3—that would leave a remainder. I was checking 4 when it struck me that 1024 is 2^{10}. I did a quick calculation.

"—9, 10. Time's up. How many?"

"Eleven. 1024 has eleven divisors."

"Correct. How did you figure it out?" Miruka occupied herself by licking chocolate off her fingers while awaiting my answer.

"From the prime factorization of 1024, which gives you two to the tenth power," I said. "If you write it out, you get this:"

$$1024 = 2^{10} = \underbrace{2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2}_{\text{ten twos}}$$

"A divisor of 1024 has to divide it evenly," I continued. "That means it has to be in the form 2^n, where n is some number from 0 to 10. So a divisor of 1024 will be one of these numbers."

Miruka nodded. "Very good. Okay, next problem. If you added up all the divisors of 1024, what would the sum—"

"Sorry," I said, abruptly standing, "but there's something I have to do before class. I'll see you later." I turned away from an obviously disgruntled Miruka and left the room.

I was already thinking of ways to find the sum of the divisors of 1024 as I headed for the roof.

2.3 The Letter

It was gloomy out. The usual lunchtime roof crowd was thin.

I took the envelope from my pocket and removed the letter. It was written on a sheet of white stationery in fountain pen with attractive handwriting.

> Hello.
>
> My name is Tetra. We went to the same junior high. I'm one year behind you. I'm writing to ask you for some advice about studying math.
>
> I've been having problems with math for years. I heard that the classes get a lot harder in high school, and I'm looking for some way to get over my "math anxiety."
>
> I know it's a lot to ask, but do you think you could spare the time to talk? I'll be in the lecture hall after school.
>
> —Tetra

Tetra? As in mono-, di-, tri-?

I was surprised to hear that we had gone to the same middle school—I didn't remember her at all. That she was having problems with math was less surprising; lots of students did, first-year students especially.

I suppose this qualified as a bona fide letter. But somehow, it wasn't as exciting as I'd hoped my first letter from a girl would be.

I read it four times.

2.4 Ambush

Classes had finished, and I was heading to the lecture hall when Miruka ambushed me, appearing out of nowhere.

"What would the sum be?" she asked.

"2047," I answered without missing a beat.

She frowned. "I gave you too much time to think about it."

"I guess. Look, I'm—"

"Headed to the library?" Her eyes flashed.

"Not today. I have to be somewhere."

"Oh, yeah? I'll give you some homework, then."

She jotted something down on a piece of paper.

Miruka's homework

Describe a method for summing the divisors of a given positive integer n.

"You want me to give you a formula in terms of n?" I asked.

"Don't strain yourself. Just the steps of a method will do."

2.5 Tetra

The lecture hall was separate from the main school building, a short walk across the school courtyard. The room was tiered, with the podium at the bottom level so students could watch a lecturer perform experiments—ideal for physics and chemistry classes.

I found Tetra standing at the back with a nervous look on her face, a notebook and pencil case clutched tightly to her chest.

"Oh, you came. Thank you so much," she said. "Um, so, I wanted to ask you some questions, but I wasn't sure how, so I asked a friend, and she said that maybe this would be a good place to, uh, meet."

Tetra and I sat down on a bench in the very back of the hall. I took the letter she had given me that morning out of my pocket.

"I read your letter, but I have to be honest. I don't remember you from junior high."

"No, of course you don't. I wouldn't expect you to."

"So how do you know about me? I didn't exactly stand out back then." It's hard to stand out when you spend all your free time in the library.

"Actually...you were kinda famous."

"If you say so..." I held up her letter. "So, about this. You said you were having trouble with math?"

"That's right. See, back in elementary school, I liked math just fine. Doing the problems, working through things—all good. But when I got to middle school, everything changed. I felt like I wasn't really getting a lot of what I was doing, you know? My math teacher told me it would only get harder in high school, so I'd need to work at it if I wanted to keep up. And I have been. But I want to do more than regurgitate what's in the books. I want to understand."

"You're worried about your grades?"

Tetra pressed her thumbnail against her lip. "No, it's not that." Her eyes darted about beneath her bangs. She reminded me of a small, nervous animal—a kitten, maybe, or a squirrel. "When I know what's going to be on a test, I'm fine. But when they start getting creative, I do worse. A lot worse."

"You follow what your teachers go over in class?"

"More or less."

"And you can do the homework?"

"Mostly. But something's not sinking in."

"All right," I nodded. "Time to understand."

2.5.1 Defining Prime Numbers

"Let's try some specifics," I began. "Do you know what a prime number is?"

"I think so," she said.

"Prove it. Give me a definition."

"Well, 5 and 7 are prime numbers..."

"Sure, but those are just examples. I want the definition."

"A prime number is, uh, a number that can only be divided by 1 and itself, right? One of my teachers made me memorize that."

"Okay. If we write that down, we get this:"

> A positive integer p is a prime number if it can be evenly divided only by 1 and p.

I showed my notebook to Tetra. "So this is your definition?"

"Yeah, that looks right."

"Close, but not quite."

"But 5 is a prime number, and it can only be divided evenly by 1 and 5."

"It works for 5. But if p was 1, according to this definition 1 would be a prime number too, since it would only be divisible by 1 and p. But the list of primes starts with 2, like this:"

$$2, 3, 5, 7, 11, 13, 17, 19, \cdots$$

"Oh right, 1 isn't a prime," Tetra replied. "I remember learning that now."

"So your definition isn't perfect, but there are a few ways to fix it. You might add a qualifier at the end, like this:"

> A positive integer p is a prime number if it can be evenly divided only by 1 and p. <u>However, 1 is not a prime</u>.

"An even better way would be to put the qualifier up front:"

> An integer p <u>that is greater than 1</u> is a prime number if it can be evenly divided only by 1 and p.

"You could also give the condition as a mathematical expression:"

> An integer $p > 1$ is a prime number if it can be evenly divided only by 1 and p.

"Those definitions make sense," Tetra looked up from my notebook. "And I know that 1 isn't a prime, but I don't get it. I mean, who says 1 can't be a prime? What difference does it make? There has to be a reason."

"A reason?" I raised an eyebrow.

"Yeah, isn't there some kind of theory or something behind all this?"

This was interesting—I didn't meet a lot of people who understood the importance of being convinced.

"That was a stupid question, wasn't it." Tetra said.

"No. No, it's a great question. The primes don't include 1 because of the uniqueness of prime factorizations."

"The uniqueness of what? You lost me."

"It's a property of numbers that says that a positive integer n will only have one prime factorization. So for 24, the only prime factorization is $2 \times 2 \times 2 \times 3$. You could write it $2 \times 2 \times 3 \times 2$ or $3 \times 2 \times 2 \times 2$ if you wanted, but they're all considered the same, because the only difference is in the order of the factors. In fact, it's so important to keep prime factorizations unique that 1 isn't included in the primes, just to protect this uniqueness."

Now it was Tetra's turn to raise an eyebrow. "You mean you can define something one way just to keep it from breaking something else?"

"Kind of a harsh way of putting it, but yeah." I tapped my pencil on the notebook. "It's more like this: Mathematicians are always on the lookout for useful concepts to help build the world of mathematics. When they find something really good, they give it a name. That's what a definition is. So you *could* define the primes to include 1 if you wanted to. But there's a difference between a *possible* definition and a *useful* one. Using your definition of primes that included 1 would mean you couldn't use the uniqueness of prime factorizations, so it wouldn't be very useful. That making sense now? The uniqueness, I mean."

"I think so."

"You think so, huh? Look, it's up to you to make sure you understand something."

"If I don't know if I understand, how can I make sure?"

"With examples. An example isn't a definition, but coming up with a good example is a great way to test one out." I wrote out a problem in my notebook and handed it to Tetra:

> Give an example showing that including 1 as a prime number would invalidate the uniqueness of prime factorizations.

"Okay," Tetra replied. "If 1 was prime, then you could factorize 24 in lots of ways. Like this:"

$$\begin{aligned} 2 \times 2 \times 2 \times 3 \\ 1 \times 2 \times 2 \times 2 \times 3 \\ 1 \times 1 \times 2 \times 2 \times 2 \times 3 \\ \vdots \end{aligned}$$

"Perfect example," I said. "See? Examples are the key to understanding." A look of relief washed over Tetra's face. "However," I continued, "instead of saying 'lots,' it would be better to say 'multiple,' or 'at least two.' Saying it that way is more. . . "

"Precise." Tetra finished.

"Exactly. 'Lots' isn't very precise, because there's no way to know how many it takes to become 'lots.'"

"All these words—definition, example, prime factorization, uniqueness. You don't know how much this helps. I didn't realize how important language was in math."

"That's a great point. Language is *extremely* important in mathematics. Math uses language in a very precise way to make sure there's no confusion. And equations are the most precise language of all."

"Equations are a language?"

"Not just any language. The language of *mathematics*—and your next lesson." I glanced around the lecture hall. "It'll be easier if I use the blackboard. C'mon."

I headed down the stairs at the center of the lecture hall. I had only taken a few steps when I heard a yelp, and Tetra came crashing into me, nearly sending both of us sprawling.

"Sorry!" Tetra said. "I tripped. On the step. Sorry!"

"It's cool," I said.

This is going to be more work than I thought.

2.5.2 Defining Absolute Values

We reached the blackboard without further incident, and I picked up a piece of chalk before turning to Tetra. "Do you know what absolute values are?"

"Yeah, I think so. The absolute value of 5 is 5, and the absolute value of -5 is 5 too. You just take away any negatives, right?"

"Well, sort of. Let's try a definition. Tell me if this looks good to you." I wrote on the board.

Definition of $|x|$, the absolute value of x:

$$|x| = \begin{cases} x & \text{if } x \geqslant 0 \\ -x & \text{if } x < 0 \end{cases}$$

"I remember having trouble with this. If getting the absolute value of x means taking away the minuses, why does a minus show up in the definition?"

"Well, 'taking away the minuses' is kinda vague in a mathematical sense. Not that I don't know what you mean. You're on the right track."

"Would it be better to say change minuses to pluses?"

"No, that's still pretty vague. Let's say we want the absolute value of $-x$:"

$$|-x|$$

"Well," Tetra said, "wouldn't you take the minus away, leaving x?"

$$|-x| = x$$

"Not quite. What if $x = -3$? How would you calculate that?"

Tetra picked up her own piece of chalk. "Let's see..."

$$\begin{aligned} |-x| &= |-(-3)| && \text{because } x = -3 \\ &= |3| && \text{because } -(-3) = 3 \\ &= 3 && \text{because } |3| = 3 \end{aligned}$$

"Right," I said. "If you use your definition, $|-x| = x$, then when $x = -3$ you would have to say that $|-x| = -3$. But in this case $|-x| = 3$. Or put another way, $|-x| = -x$."

Tetra stared at the board. "I see. Since there's no sign in front of the x, I never thought of x being a negative number like -3. But that's the whole point of using a letter like x, isn't it—it lets you define something without giving all sorts of specific examples."

"That's right," I said. "Just saying 'take off the minuses' isn't good enough. You have to be strict with yourself—you have to think about all the possibilities. No cutting corners."

Tetra nodded slowly. "Guess I'm gonna have to get used to that." She slumped into a chair and started fiddling with the corner of her notebook. "I was just wasting time in junior high," she said.

I waited quietly for her to continue.

"Not that I didn't study. But I wasn't looking at the definitions and equations the right way. I wasn't strict enough. I was too... sloppy." Tetra let out a long sigh.

"That's in the past," I said.

"Huh?"

"Just do things right from here on out."

She sat up, eyes wide.

"You're right. I can't change the past, but I can change myself."

I smiled. "Glad I could help with the breakthrough, but we should probably call it a day. It's getting dark out. We'll pick this up next time."

"Next time?"

"I'm usually in the library after school. If you have any more questions, you know where to find me."

2.6 Beneath an Umbrella

Outside the lecture hall, Tetra stopped and looked up at the sky. Clouds hung low and grey, and it had started to rain.

"Figures," she grumbled.

"No umbrella?" I asked.

"I was running late this morning. Guess I forgot it. I even watched the weather and everything!" She shrugged. "Well, it's not raining hard. I'll be okay if I run."

"You'll be soaked by the time you get to the train station. C'mon, we're going the same way. My umbrella's big enough for both of us."

She smiled. "Thanks!"

I'd never shared an umbrella with a girl before. It was a bit awkward at first, but I matched her pace and managed to keep from tripping or jabbing her with an elbow. We walked slowly through the soft spring shower. The road was quiet, the bustle of the town lost in the rain.

It was cool to have someone younger who looked up to me, and I found myself surprised by just how much I had enjoyed talking with her. She was so easy to read—when she understood something, and when she didn't.

"So how'd you know?" she asked.

"Know what?"

"How'd you know what was giving me trouble?"

"Oh, that. Well, a lot of the stuff we talked about today—prime numbers, absolute values, all that—that's stuff I wondered about too, back in the day. When I'm studying math and I don't understand something, it bugs me. I'll think about it for weeks, I'll read about it, and then suddenly I'll just *get it.* And the feeling when it happens... After you've felt that a few times you can't help but like math. And then you start getting better at it, and—oh, we should turn here."

"That's not the way to the station."

"It's a lot quicker if you cut through this neighborhood."

"Oh..."

"Yeah, it's a great shortcut."

Tetra slowed her pace to a crawl and I found myself having a hard time matching her speed the rest of the way.

The rain was still falling when we reached the station.

"I think I'm gonna hit the book store," I said. "Guess I'll see you tomorrow." I started to leave. "Oh, here," I offered her my umbrella. "Why don't you take this."

"You're going? Oh. Okay, well. . . thanks for all the help. It really means a lot to me." She bowed deeply.

I nodded and darted for the bookstore.

Tetra called after me. "And thanks for the umbrella!"

2.7 Burning the Midnight Oil

That night I sat in my room, recalling my conversation with Tetra. She had been so sincere, so enthusiastic. She definitely had potential. I hoped that she would learn to enjoy math.

When I talked to Tetra, I slipped into teacher mode. Talking with Miruka was a very different thing. With Miruka, I had to scramble to keep up. If anything, she was the one teaching me. I remembered the homework she had given me; without a doubt the first time I'd gotten homework from another student.

Miruka's homework

Describe a method for summing the divisors of a given positive integer n.

I knew that I could always solve the problem by finding all the divisors of n and adding them together, but that felt like a cheat. I wondered if I couldn't find a better way, and the prime factorization of n looked like a good place to start.

I thought back on the problem we worked on at lunch, for $1024 = 2^{10}$. Maybe there was some way to generalize this, like writing n as a power of a prime:

$$n = p^m \quad \text{for prime number } p, \text{ positive integer } m$$

If $n = 1024$, then for this equation I'd have $p = 2$ and $m = 10$. If I wanted to list all the divisors of n like I did for 1024, then it would be something like this:

$$1, p, p^2, p^3, \cdots, p^m$$

So for $n = p^m$, I could find the sum of the divisors by adding them up:

$$(\text{sum of divisors of } n) = 1 + p + p^2 + p^3 + \cdots + p^m$$

That would be the answer for a positive integer n that could be written in the form $n = p^m$, at least. I pushed on to see if I couldn't generalize it for other numbers too. It shouldn't be too hard; all I needed to do was generalize prime factor decomposition.

One way to write the prime factor decomposition for a positive integer n would be to take $p, q, r, \cdots$ as primes and $a, b, c, \cdots$ as positive integers and write it like this:

$$n = p^a \times q^b \times r^c \times \cdots \times \text{whoa!}$$

Hang on. This won't work using just letters. If I went through $a, b, c, \cdots$ they'd eventually run into $p, q, r, \cdots$ and that would really confuse things.

I wanted to write an expression that looked something like $2^3 \times 3^1 \times 7^4 \times \cdots \times 13^3$, the product of a bunch of terms in the form $\text{prime}^{\text{integer}}$. So I could write the primes as $p_0, p_1, p_2, \cdots, p_m$ and the exponents as $a_0, a_1, a_2, \cdots, a_m$. Adding all of those subscripts might make things look a bit more complicated, but at least it would let me generalize. It would also let me use $m+1$ to mean the number of prime factors in the prime factor decomposition of n. I started rewriting.

Now given a positive integer n, I could generalize its prime factor decomposition:

$$n = p_0^{a_0} \times p_1^{a_1} \times p_2^{a_2} \times \cdots \times p_m^{a_m},$$

where $p_0, p_1, p_2, \cdots, p_m$ are primes and $a_0, a_1, a_2, \cdots, a_m$ are positive integers. When n was in this form, then a divisor of n would look like this:

$$p_0^{b_0} \times p_1^{b_1} \times p_2^{b_2} \times \cdots \times p_m^{b_m},$$

where $b_0, b_1, b_2, \cdots, b_m$ was an integer:

$$\begin{aligned}
b_0 &= \text{one of } 0, 1, 2, 3, \cdots, a_0 \\
b_1 &= \text{one of } 0, 1, 2, 3, \cdots, a_1 \\
b_2 &= \text{one of } 0, 1, 2, 3, \cdots, a_2 \\
&\vdots \\
b_m &= \text{one of } 0, 1, 2, 3, \cdots, a_m
\end{aligned}$$

I looked back at what I had written, surprised at how messy it was to write it out precisely. All I wanted to say was that, to write a divisor, you just leave the prime factors as they are, and move through the exponents $0, 1, 2, \cdots$ for each one. But generalizing this took an alphabet soup's worth of symbols.

With things generalized to this extent, I figured the rest would be easy. To find the sum of the divisors I just had to add all of these up.

$$\begin{aligned}
(\text{sum of divisors of } n) = {} & 1 + p_0 + p_0^2 + p_0^3 + \cdots + p_0^{a_0} \\
& + 1 + p_1 + p_1^2 + p_1^3 + \cdots + p_1^{a_1} \\
& + 1 + p_2 + p_2^2 + p_2^3 + \cdots + p_2^{a_2} \\
& + \cdots \\
& + 1 + p_m + p_m^2 + p_m^3 + \cdots + p_m^{a_m}
\end{aligned}$$

I paused, realizing that what I had written was wrong. This wasn't the sum of all the divisors, it was the sum of just those divisors that can be described as a power of a prime factor. I had said the form of a divisor was this:

$$p_0^{b_0} \times p_1^{b_1} \times p_2^{b_2} \times \cdots \times p_m^{b_m}$$

So I had to find all the combinations of powers of prime factors, multiply those together, and add them up. I found this easier to write as an equation than to put into words, so that's what I did.

My answer to Miruka's homework

Write the prime factorization of the positive integer n as follows:

$$n = {p_0}^{a_0} \times {p_1}^{a_1} \times {p_2}^{a_2} \times \cdots \times {p_m}^{a_m},$$

where $p_0, p_1, p_2, \cdots, p_m$ are prime numbers and $a_0, a_1, a_2, \cdots, a_m$ are positive integers. Then the sum of the divisors of n is as follows:

$$\begin{aligned}(\text{sum of divisors of } n) = & \ (1 + p_0 + p_0^2 + p_0^3 + \cdots + p_0^{a_0}) \\ & \times (1 + p_1 + p_1^2 + p_1^3 + \cdots + p_1^{a_1}) \\ & \times (1 + p_2 + p_2^2 + p_2^3 + \cdots + p_2^{a_2}) \\ & \times \cdots \\ & \times (1 + p_m + p_m^2 + p_m^3 + \cdots + p_m^{a_m})\end{aligned}$$

I went to bed wondering if there wasn't a cleaner way to write this... and whether I was even right at all.

2.8 Miruka's Answer

"Well, it's right," Miruka said the next day, "but it's kind of a mess."

Blunt as ever, I thought, but what I said was, "Is there some way to make it simpler?"

"Yes," she immediately replied. "First, you can use this for the long sums." Miruka started writing in my notebook as she talked. "Assuming that $1 - x \neq 0\ldots$"

$$1 + x + x^2 + x^3 + \cdots + x^n = \frac{1 - x^{n+1}}{1 - x}$$

"Oh, of course. The formula for the sum of a geometric progression."

Miruka jotted down the proof. *Show off.*

$$\begin{aligned} 1 - x^{n+1} &= 1 - x^{n+1} && \text{equal sides} \\ (1 - x)(1 + x + x^2 + x^3 + \cdots + x^n) &= 1 - x^{n+1} && \text{factor left side} \\ 1 + x + x^2 + x^3 + \cdots + x^n &= \frac{1 - x^{n+1}}{1 - x} && \text{divide by } 1 - x \end{aligned}$$

"You can use that to turn all your sums of powers into fractions," she continued. "You should also use $\prod$ to tidy up the multiplication." She wrote the symbol large on the page.

"That's a capital π, right?"

"Right. This one doesn't have anything to do with circles, though. $\prod$ works like Σ does, but for multiplication. Σ is a capital Greek 'S' for 'sum,' and $\prod$ is a capital Greek 'P' for 'product.' If you wanted to write out a definition for it..."

Definition of the Sigma operator

$$\sum_{k=0}^{m} f(k) = f(0) + f(1) + f(2) + f(3) + \cdots + f(m)$$

Definition of the Pi operator

$$\prod_{k=0}^{m} f(k) = f(0) \times f(1) \times f(2) \times f(3) \times \cdots \times f(m)$$

"Now, check out how much cleaner things are when you use Π."

Miruka's answer

Write the prime factorization of the positive integer n as follows:

$$n = \prod_{k=0}^{m} p_k^{a_k},$$

where p_k is a prime number and a_k is a positive integer. Then the sum of the divisors of n is as follows:

$$(\text{sum of divisors of } n) = \prod_{k=0}^{m} \frac{1 - p_k^{a_k+1}}{1 - p_k}.$$

"Okay," I said. "Lots of letters in there, but it is shorter. By the way, you going to the library today?"

"Nope." Miruka shook her head. "I'm off to practice with Ay-Ay. She says she has a new piece ready."

2.9 Math by the Letters

I was working on some equations when Tetra came up to me with a smile and an open notebook.

"Look what I did! I copied all the definitions out of my math book from last year, and made my own example for every one!"

"All in one night? That's dedication."

"Oh no, I love doing stuff like this. And I thought of something when I was going through my old textbook. Maybe the difference between simple and advanced math is that, in advanced math, you use letters in the equations."

2.9.1 Equations and Identities

I nodded. "Right, and since you brought up using letters in equations, let's talk a little bit about equations and identities. You've seen equations like this:"

$$x - 1 = 0$$

"Sure. $x = 1$."

"How about this one?"

$$2(x - 1) = 2x - 2$$

She frowned. "I think I can solve that if I clean it up a little:"

$2(x - 1) = 2x - 2$	the problem
$2x - 2 = 2x - 2$	expand the left side
$2x - 2x - 2 + 2 = 0$	move the right side terms to the left
$0 = 0$	simplify the left side

"Huh? I ended up with $0 = 0$."

"That's right, because $2(x - 1) = 2x - 2$ isn't an equation, it's an identity. See how when you expanded the $2(x - 1)$ on the left side, you got $2x - 2$ on the other? They're *identical*, right? To be precise, this is an identity in x. That means that no matter what x is, the statement will be true."

"So identities are different from equations?"

"Uh huh. An equation is a statement that's true when you replace the xs with a *certain* number. An identity is a statement that's true when you replace the xs with *any* number. When you're doing a problem that deals with an equation, you're probably trying to find the value of x that makes the statement true. When you're doing a problem that deals with an identity, you're probably trying to show that any value of x will work. Do that, and you've proven the identity."

"I get it. I guess I've always known about identities, I just never thought of them as being so different."

"Most people don't. But you use them all the time. Almost all the formulas you learn outside of math are actually identities."

"How can you tell the difference?"

"You have to look at the context and ask yourself what the person who wrote it intended it to be."

"I'm not sure I follow."

"Well, for example, if you want to change the form of a statement, you use identities. Here, look at this:"

$$\begin{aligned}(x+1)(x-1) &= (x+1)\cdot x-(x+1)\cdot 1\\ &= x\cdot x+1\cdot x-(x+1)\cdot 1\\ &= x\cdot x+1\cdot x-x\cdot 1-1\cdot 1\\ &= x^2+x-x-1\\ &= x^2-1\end{aligned}$$

"See the equals signs in each line? They're forming a chain of identities. You can follow the chain, checking everything out step by step, until you end up with this:"

$$(x+1)(x-1) = x^2-1$$

"Okay."

"Chains of identities like this give you a slow-motion replay of how a statement transforms from one thing into another. Don't freak out because there's a bunch of statements. Just follow them along, one at a time. Now take a look at this:"

$$\begin{aligned}x^2-5x+6 &= (x-2)(x-3)\\ &= 0\end{aligned}$$

"The first equals sign there is creating an identity. It's telling you that no matter what you stick into this x here, $x^2-5x+6 = (x-2)(x-3)$. But that second equals sign is creating an equation. It's saying you don't have to solve $x^2-5x+6=0$ for x, you can just solve its identity, $(x-2)(x-3)=0$, instead."

"Not bad for two lines."

I nodded. "There's one more kind of equality besides equations and identities that you should know about: definitions. When you have a really complex statement, definitions let you name part of it to simplify things. You use an equals sign for this, but it doesn't mean you have to solve for anything, like with an equation, or prove anything, like with an identity. You just use them in whatever way's convenient."

"Can you give me an example?"

"Well, say you're adding together two numbers, alpha and beta. You could name them—in other words define them as—'s' like this:"

$$s = \alpha + \beta \quad \text{an example definition}$$

Tetra's hand shot up. "Question!"

"This isn't class, Tetra. You don't have to raise your hand."

She lowered her hand. "But I'm confused. Why did you name it 's'?"

"It doesn't really matter what you name it. You can use s, t, whatever you want. Then once you've said, okay, from now on $s = \alpha + \beta$, you can just write s instead of having to write $\alpha + \beta$ every time. Learn to define things, and you'll be able to write math that's easier to read *and* understand."

"So what are α and β then?"

"Well, they could be letters that you defined somewhere else. When you define something like $s = \alpha + \beta$, that usually means you're using the letter on the left side of the equals sign to name the expression on the right. So here, you'd be using s as the name of something that you made out of α and β."

"And you can name them anything you want, right?"

"Basically, yeah. Except that you shouldn't use a name that you've already used to define something else. Like, if you defined $s = \alpha + \beta$ in one place, and then turned around and redefined it as $s = \alpha\beta$, you'd start to lose your audience."

"Yeah, I can see that."

"There are also some generally accepted definitions, like using π to mean the ratio of the circumference of a circle to its diameter, or i to represent the imaginary unit, so it would be kind of weird to use those names for something else. Anyway, if you're reading through a math problem and you see a new letter popping up, don't panic, just think to yourself, 'oh, this must be a definition.' If you're reading math and it says something like 'define s as $\alpha + \beta$' or 'let s be $\alpha + \beta$' you're looking at a definition."

"Got it."

I put down my pencil. "Here's an idea. Next time you're going through your book, try looking for mathematical statements with letters in them and asking yourself if they're equations, identities, definitions, or something else altogether."

Tetra nodded enthusiastically.

"You know," I told her, "every mathematical statement you find in your textbook was written to express a thought. Just remember," I said, pausing for effect, "there's always somebody behind the math, sending us a message."

2.9.2 The Forms of Sums and Products

"Oh, one more important thing," I told her. "You should always pay attention to the overall form of a mathematical expression."

"What's that mean?"

"Take a look at this statement, for example. An equation, right?"

$$(x-\alpha)(x-\beta)=0$$

"The expression on the left side of the equals sign is telling you to multiply. In other words, it's in multiplicative form. The things that are being multiplied together are called factors."

$$\underbrace{(x-\alpha)}_{\text{factor}}\underbrace{(x-\beta)}_{\text{factor}}=0$$

"The same 'factor' in prime factorization?" Tetra asked.

"Sure. Factorizing something means breaking it down into a multiplicative form. Prime factorization means breaking it down into a multiplicative form where all the factors are prime numbers. Oh, and most people leave out the multiplication sign when multiplying things. So all of these are the same equation, just written different ways:"

$$\begin{aligned}(x-\alpha)\times(x-\beta)&=0 &&\text{using a } \times \text{ sign}\\ (x-\alpha)\cdot(x-\beta)&=0 &&\text{using a } \cdot \text{ sign}\\ (x-\alpha)(x-\beta)&=0 &&\text{using nothing}\end{aligned}$$

"Okay," she said.

"Now," I added, "for $(x-\alpha)(x-\beta)=0$ we know that at least one of the two factors has to equal zero. We can say that because it's in multiplicative form."

"So...if we multiply two things together and the result is zero, one of the factors has to be zero. That makes sense."

"Well, it's better to say that *at least* one of them has to be zero. Because they both might be, right?"

"Okay, *at least* one of them. This is that precise mathematical language thing we talked about yesterday, isn't it."

"Right. So anyway, since we know that at least one of the factors is zero, do you see how this equation is true when $x - \alpha = 0$ or $x - \beta = 0$? Another way to say this is that $x = \alpha, \beta$ is a solution to this multiplicative form equation."

"I follow."

"Okay. So let's see what happens when we expand $(x-\alpha)(x-\beta)$:"

$$(x - \alpha)(x - \beta) = x^2 - \alpha x - \beta x + \alpha\beta$$

"By the way," I asked her, "do you think this is an equation?"

"Nope!" Tetra replied quickly. "It's an identity!"

"Not bad. Okay, 'expanding something' means changing products into sums. On the left side there are two factors being *multiplied* together, and on the right side there are four terms being *added* together."

"Sorry, terms?"

"Yeah, when you add things together, you call them terms. Here, let me show you a diagram with everything labeled:"

$$\underbrace{(x - \alpha)}_{\text{factor}}\underbrace{(x - \beta)}_{\text{factor}} \quad \overset{\xrightarrow{\text{expand}}}{\underset{\xleftarrow[\text{factorize}]{}}{=}} \quad \underbrace{(x^2)}_{\text{term}} + \underbrace{(-\alpha x)}_{\text{term}} + \underbrace{(-\beta x)}_{\text{term}} + \underbrace{(\alpha\beta)}_{\text{term}}$$

"We can still do some cleanup on this expression," I continued. "It's a bit of a mess as it is:"

$$x^2 - \alpha x - \beta x + \alpha\beta$$

"Well," Tetra said, "we could take the things that have an x in them, like $-\alpha x$ and $-\beta x$..."

"Try to call them 'terms,' not 'things,' okay?" I said. "Also, terms like $-\alpha x$ and $-\beta x$ that only have one x should be called 'first degree terms of x,' or simply 'first degree terms.'"

"Okay..." Tetra scratched her head. "How 'bout we bring together the first degree terms of x. Like this:"

$$x^2 + \underbrace{(-\alpha - \beta)x}_{\text{first degree terms}} + \alpha\beta$$

"Exactly. That's a good explanation of what to do with terms, but normally you would go one more step and bring the minus sign to the outside of the parentheses:"

$$x^2 - (\alpha + \beta)x + \alpha\beta$$

"You've probably heard of that as 'combining like terms.'"

She frowned. "Heard of it, yes. Thought about it, no."

"A quick quiz, then. Is this an identity or an equality?"

$$(x - \alpha)(x - \beta) = x^2 - (\alpha + \beta)x + \alpha\beta$$

"All we've done is expand and combine like terms, right? So this should be true for any value of x... which makes it an identity!"

"Very good! Moving on, then. We started out talking about this equation, which is in multiplicative form:"

$$(x - \alpha)(x - \beta) = 0 \qquad \text{equation in multiplicative form}$$

"Using the identity that we just created, we can rewrite the equation. This is called an equation in additive form:"

$$x^2 - (\alpha + \beta)x + \alpha\beta = 0 \qquad \text{equation in additive form}$$

"These equations are in different forms, but they're the same equation. All we've done is use an identity to change the form of the left side."

"Got it."

"When we looked at the multiplicative form, we said that the solution to the equation was $x = \alpha, \beta$. That means the solution to the equation in additive form must also be $x = \alpha, \beta$. After all, they're the same equation."

$$\begin{array}{rl} (x-\alpha)(x-\beta) = 0 & \text{equation in multiplicative form} \\ \Updownarrow & \text{same equations, same solutions} \\ x^2 - (\alpha+\beta)x + \alpha\beta = 0 & \text{equation in additive form} \end{array}$$

"You can use this to solve some simple second degree equations just by looking at them. Here, take a look at these two. Pretty similar, aren't they?"

$$\begin{array}{rl} x^2 - (\alpha+\beta)x + \alpha\beta = 0 & \quad (\text{solution: } x = \alpha, \beta) \\ x^2 - 5x + 6 = 0 & \end{array}$$

"Well..." She paused for a moment. "Oh, I see! You can just think of the 5 as being $\alpha + \beta$, and the 6 as $\alpha\beta$."

"Exactly. So to solve $x^2 - 5x + 6 = 0$, all you have to do is think of two numbers that equal 5 when you add them, or 6 when you multiply them. That would be $x = 2, 3$, right?"

"Makes sense."

"Mathematical expressions come in all sorts of forms. Multiplicative and additive are just two of the possibilities. Remember that solving equations like $\langle \text{additive form} \rangle = 0$ can be tough, but problems like $\langle \text{multiplicative form} \rangle = 0$ are super simple."

"Huh, it's like putting equations in multiplicative form is a way of solving them, isn't it," Tetra said. "You know, I think I'm getting the hang of this."

2.10 Who's Behind the Math?

"I wish my teachers taught me as well as you do," Tetra said.

I grinned. "It's a lot easier one-on-one. If I lose you, you can slow me down and ask questions. You could always try that in class sometimes."

Tetra pondered that for a moment. "What if I'm studying something and there's no one around to ask?" she asked.

"If I don't get something after a careful reading, I mark the page and move on. After a while, I'll come back to that page and read it one more time. If I still don't get it, I move on again. Sometimes I'll switch to a different book—but I keep going back to the part I didn't understand. Once, I came across an expansion of an equation that I just couldn't follow. After agonizing over it for four days, I decided there was no way it could be right, so I contacted the publisher. Turns out it was a misprint."

"Nice catch!" Tetra shook her head. "Guess it pays to keep at it."

"Well, math takes time. I mean, there's so much history to it. When you're reading math, you're trying to relive the work of countless mathematicians. Trace through the development of a formula, and you might be following *centuries* of work. With depth like that, it's not enough just to read. You have to become a mathematician yourself."

"Sounds like a tall order."

"Well, it's not like you have to get a PhD, but when you're reading math, you do have to make an effort to get into it. Don't just read it, write it out. That's the only way to be sure you really understand."

Tetra gave a slight nod. "It kind of got to me, what you said about equations being a language, and that there's somebody on the other side of the equation trying to send a message." She looked off into the distance, her words coming faster. "Maybe it's only my teacher, or the author of the textbook, but I can imagine it being a mathematician from hundreds of years ago too. It kind of makes math *real*, if you know what I mean."

She looked away and smiled. "I think I might love you—" She looked back at me, then her eyes went wide as she realized what she had just said.

I cocked an eyebrow.

"—Teaching me!" She blurted, a few seconds too late.

I looked around nervously.

Tetra's face was burning red. "Math," she whispered. "I love you teaching me math."

Index

A
absolute inequality, 71
absolute value, 18, 52
additive form, 32
arbitrarily close, 89
argument, 41
arithmetic mean, 76
average, 72

B
Basel problem, 209, 212, 216, 259
binomial theorem, 109, 234

C
Catalan numbers, 138, 202, 233
closed expression, 54
coefficient, 53
combination, 111, 234
combining like terms, 32
common ratio, 52
complex number, 38
condition, 73
continuous, 88
convolution, 138, 174
cosine function, 193

D
definition, 17, 28
 of absolute value, 18
 of prime numbers, 15
derivative, 87, 133, 165
difference, 90
 of squares, 49
difference equation, 96
differential operator, 88
differentiation, 190, 191, 261
 of trig functions, 195
 rules, 192, 194
discrete, 89, 165
divergance, 159, 162
divisors, 12
double angle formulas, 36

E
e, 95
element ($\in$), 148

equation
  difference, 96, 165, 167
  quadratic, 63, 121
  second degree, 33, 60
  simultaneous, 61
  system of, 62
  versus identity, 26
Euler, Leonhard, 180, 217, 234
exists (∃), 151
exponent
  fractional, 78
  negative, 166

F
factor, 30
factorization, 30, 60, 207, 211, 218
falling factorial, 93, 112, 134, 165
Fibonacci numbers, 244
Fibonacci sequence, 6, 55, 56, 59, 202, 233
first degree term, 32
for all (∀), 148
formula, 76
  for a geometric series, 161, 175, 241, 257
function
  exponential, 95, 164
  generating, 53, 117
  higher order, 88
fundamental theorem of algebra, 206

G
general term, 7, 54, 117, 233, 236
  for the Fibonacci sequence, 62
generalization, 22, 37, 51, 108, 109, 188
generating function, 53, 55, 57, 117, 232, 236, 253
geometric mean, 76
geometric progression, 52, 175
geometric series, 52, 178
given, 75

I
i, 38, 145
identity, 71
  versus equation, 26, 186
imaginary number, 38, 78, 145
inequality, 157
  absolute, 71
  of arithmetic and geometric means, 76, 77
infinity, 141
instantaneous rate of change, 87
integers, 71

L
limit, 52, 142, 218
logarithm, 165, 254

M
mathematical induction, 244
max-min table, 261
mean
  arithmetic, 76
  geometric, 76
memorization, 7, 76
multiplicative form, 30

N

number
- harmonic, 163, 168, 260
- imaginary, 78, 145
- integer, 71
- irrational, 96
- natural, 148
- negative, 73
- prime, 15, 176
- real, 70, 71
- square, 70

O

operator
- difference, 90, 167
- differential, 88, 165
- product, 80
- summation, 80

P

partition number, 221, 236, 253
- upper limit, 243

Π, 25, 80

π, 7, 217

polynomial, 187
- infinite dimensional, 187

power laws, 57

power series, 53, 117, 131, 187, 191, 201

precision
- in language, 146

prime factorization, 22, 26, 30

product
- formal, 174
- of a sum, 49
- of infinite series, 175
- of infinite sums, 177
- of sums, 119

proof by contradiction, 176

proposition, 145

Q

quadratic equation, 63

quadratic formula, 121

R

real number, 70, 71

recurrence relation, 57, 105, 233

rotation, 35

S

sequence, 53
- infinite, 132
- problems, 9

series
- geometric, 52
- harmonic, 162, 179, 204
- infinite, 52, 59, 142, 200

Σ, 25, 80, 107, 114, 254

sine function, 188, 190, 210, 218

square number, 70

square root, 73, 78

subscripts, 22, 144

sum
- finite, 144
- of a geometric progression, 25
- of powers, 53
- of products, 118
- partial, 52, 142, 200

T

Taylor series, 196, 198, 200, 214, 255

U

uniqueness of prime factorizations, 16, 176, 178
unit circle, 40
upper limit, 243, 253

V

variable
naming, 29, 69, 71, 145, 188

Z

zeta function, 155, 162, 204, 209

Other works by Hiroshi Yuki

(in Japanese)

- *The Essence of C Programming*, Softbank, 1993 (revised 1996)
- *C Programming Lessons, Introduction*, Softbank, 1994 (Second edition, 1998)
- *C Programming Lessons, Grammar*, Softbank, 1995
- *An Introduction to CGI with Perl, Basics*, Softbank Publishing, 1998
- *An Introduction to CGI with Perl, Applications*, Softbank Publishing, 1998
- *Java Programming Lessons (Vols. I & II)*, Softbank Publishing, 1999 (revised 2003)
- *Perl Programming Lessons, Basics*, Softbank Publishing, 2001
- *Learning Design Patterns with Java*, Softbank Publishing, 2001 (revised and expanded, 2004)
- *Learning Design Patterns with Java, Multithreading Edition*, Softbank Publishing, 2002

- *Hiroshi Yuki's Perl Quizzes*, Softbank Publishing, 2002
- *Introduction to Cryptography Technology*, Softbank Publishing, 2003
- *Hiroshi Yuki's Introduction to Wikis*, Impress, 2004
- *Math for Programmers*, Softbank Publishing, 2005
- *Java Programming Lessons, Revised and Expanded (Vols. I & II)*, Softbank Creative, 2005
- *Learning Design Patterns with Java, Multithreading Edition, Revised Second Edition*, Softbank Creative, 2006
- *Revised C Programming Lessons, Introduction*, Softbank Creative, 2006
- *Revised C Programming Lessons, Grammar*, Softbank Creative, 2006
- *Revised Perl Programming Lessons, Basics*, Softbank Creative, 2006
- *Introduction to Refactoring with Java*, Softbank Creative, 2007
- *Math Girls / Fermat's Last Theorem*, Softbank Creative, 2008
- *Revised Introduction to Cryptography Technology*, Softbank Creative, 2008
- *Math Girls Comic (Vols. I & II)*, Media Factory, 2009
- *Math Girls / Gödel's Incompleteness Theorems*, Softbank Creative, 2009
- *Math Girls / Randomized Algorithms*, Softbank Creative, 2011